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A higher integrability result for a parameter estimation problem

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An Applied Question

How can we estimate the thermal conductivity from temperature measurements?
For a model-based parameter estimation of thermal conductivity $k = k(x)$ distributed in a smooth spatial domain $\Omega \subset \mathbb{R}^d$, let us assume that temperature $u = u(x)$ satisfies the stationary heat equation

$$\left.\begin{aligned} -\operatorname{div}(k\nabla u) &= f \quad \text{in } \Omega \\ u &= 0 \quad \text{on } \partial\Omega \end{aligned}\right\} \qquad (*)$$

for a known heat source $f = f(x)$. Given temperature measurements $\hat{y} \in \mathbb{R}^N$ in N parts of Ω, the task is to find $\hat{k}$ such that the corresponding solution $\hat{u}$ of (*) has in these N parts of Ω averaged projected values $P\hat{u}$ such that $\|P\hat{u} - \hat{y}\|$ becomes minimal.

However, this minimization problem is ill-posed, as k is not uniquely determined by (*) and N averaged values of u. Moreover, even if u is known everywhere in Ω so that (*) can be considered as first order hyperbolic equation for k, then k is still not unique as it is not prescribed along some non-characteristic hypersurface in Ω.

Therefore, in correspondence with the physical fact that k often is piecewise constant due to sharp interfaces between subregions of Ω filled by materials with varying thermal conductivity, i.e. ∇k is sparse, we regularize the minimization problem by the total variation $TV(k) := \int_\Omega \nabla k$ of k and obtain …

The Corresponding Regularized Optimization Problem

Given $f \in L^2(\Omega)$, $0 < r < R < +\infty$, $\gamma > 0$, a continuous linear operator $P : W_0^{1,2}(\Omega) \to \mathbb{R}^N$ and $\hat{y} \in \mathbb{R}^N$, find $k \in BV(\Omega) := \{k \in L^1(\Omega) \,|\, \nabla k \text{ is a finite Radon measure}\}$ such that

$$\begin{aligned} &J(k) := \|Pu - \hat{y}\|_{\mathbb{R}^N} + \gamma TV(k) = \min! \\ &s.t. -\operatorname{div}(k\nabla u) = f \text{ in } \Omega\,, \qquad u = 0 \text{ on } \partial\Omega\,, \qquad r \leq k \leq R \text{ in } \Omega\,. \end{aligned} \qquad (**)$$

Colonius and Kunisch [1] wrote one of the first articles about a similar regularized parameter estimation problem, but there instead of $BV(\Omega)$ and total variation TV a space of continuous functions and a regularizing norm is used which guarantee existence of strong solutions $u \in W^{2,2}(\Omega)$ of (*) (and also newer articles about such optimization problems always seem to assume existence of strong solutions). However, already for dimension $d = 2$ there exist symmetric matrix-valued coefficients $k \in W^{1,d}(\Omega) \not\subset VMO(\Omega)$ satisfying a bound $0 < r \leq k \leq R < +\infty$ such that the weak solution $u \in W_0^{1,2}(\Omega)$ of (*) satisfies $u \in W^{2,p}(\Omega)$ for $p < 2$, but $u \notin W^{2,2}(\Omega)$ (see Cruz-Uribe, Moen and Rodney [2] for a discussion). In the case of less regular non-continuous coefficients, the situation even becomes worse, and in the case of elliptic systems instead of scalar elliptic equations only partial regularity is available. However, to prove existence of minimizers of (**) you need at least higher integrability of weak solutions $u \in W_0^{1,2}(\Omega)$ of (*) and have to overcome some mathematical difficulties.

Mathematical Difficulties

1. $BV(\Omega)$ is not a separable reflexive Banach space, but has a separable predual. Moreover, $BV(\Omega) \subset L^q(\Omega)$ continuously (resp. compactly) for $1 \leq q \leq \frac{d}{d-1}$ ($1 \leq q < \frac{d}{d-1}$), but $BV(\Omega) \not\subset C(\overline{\Omega})$.
2. For a minimizing sequence k_n, after having obtained subsequences $k_n \overset{*}{\rightharpoonup} \hat{k}$ in $BV(\Omega)$ and $u_{k_n} \rightharpoonup \hat{u}$ in $W_0^{1,2}(\Omega) \cap W^{1,s}(\Omega)$, the estimate
$$\int_\Omega (k_n - \hat{k})\nabla u \cdot \nabla v\, dx \leq \|k_n - k\|_{L^q}\|\nabla u\|_{L^s}\|\nabla v\|_{L^r}$$
with $s > 2$, $r > 2$, sufficiently large is needed to identify the limit $\hat{u}$ with the solution of (*) to coefficients $\hat{k}$, i.e. higher integrability of weak solutions $u \in W_0^{1,2}(\Omega)$ of (*) has to be known a priori.

Generalizations and Open Questions

In [3] we discuss existence and stability for abstract minimization problems $J(k) = \min!$ and general elliptic systems $A_k u = f$. It seems to be an open question how these results can be generalized to nonlinear operators A_k.

A higher integrability result

To obtain higher integrability of ∇u, in [3] we estimate the change of the norm $\|k\nabla u\|_q$ while q increases from 2 to a sufficiently large value $s > 2$: Define $\langle A_k u, v\rangle := \int_\Omega k\nabla u \cdot \nabla v\, dx$, then

$$\begin{aligned} &\frac{d}{dq}\|k\nabla u\|_q + \frac{1}{q}\frac{1}{\|k\nabla u\|_q^{q-1}}\langle A_k u - f, \operatorname{div}((k\nabla u)^{q-1})\rangle \\ = \;&\frac{1}{q^2}\|k\nabla u\|_q\left(\int_\Omega \frac{|k\nabla u|^q}{\|k\nabla u\|_q^q}\ln\left(\frac{|k\nabla u|^q}{\|k\nabla u\|_q^q}\right)dx - \frac{2}{\|k\nabla u\|_q^q}\int_\Omega(\tilde{q}-1)|\nabla((k\nabla u)^{\frac{q}{2}})|^2\,dx\right) \\ &+ \frac{1}{q}\frac{1}{\|k\nabla u\|_q^{q-1}}\langle \nabla f, (k\nabla u)^{q-1}\rangle \end{aligned}$$

Thus, by substituting $w := |k\nabla u|^{\frac{q}{2}}$ and estimating $-(q-1) \leq -(\tilde{q}-1)$, the term in brackets is identical with the left hand side of Gagliardo-Nirenberg inequalities in their parameteric form

$$\int \frac{|w|^2}{\|w\|_2^2}\ln\left(\frac{|w|^2}{\|w\|_2^2}\right)dx - \mu\frac{\|\nabla w\|_2^2}{\|w\|_2^2} \leq \frac{2}{2 - 2^2/2^*}\ln\left(\frac{2C_{n,2,2}^2}{e(2-2^2/2^*)\mu}\right)$$

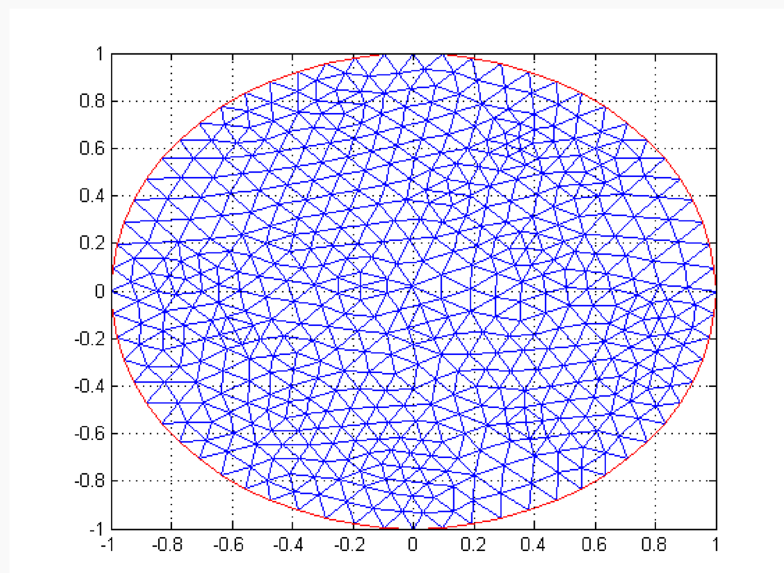
for $\mu := 2$. Hence, we obtain

$$\frac{d}{dq}\|k\nabla u\|_q \leq \frac{1}{q^2}\|k\nabla u\|_q\frac{n}{2}\ln\left(\frac{nC_{n,2,2}^2}{4e}\right) + \frac{1}{q}\|\nabla f\|_q$$

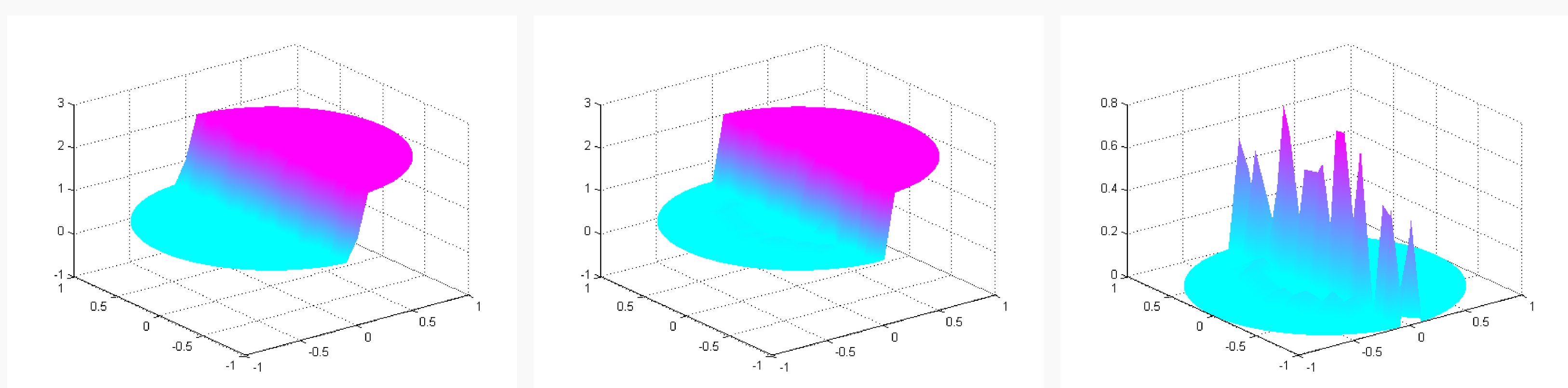
and from this for every $q \leq \infty$ a bound of $k\nabla u$ in L^q by the L^q-norm of ∇f. Finally, we use the lower bound $r \leq k$ to obtain a bound of ∇u in L^q.

Numerical Experiments

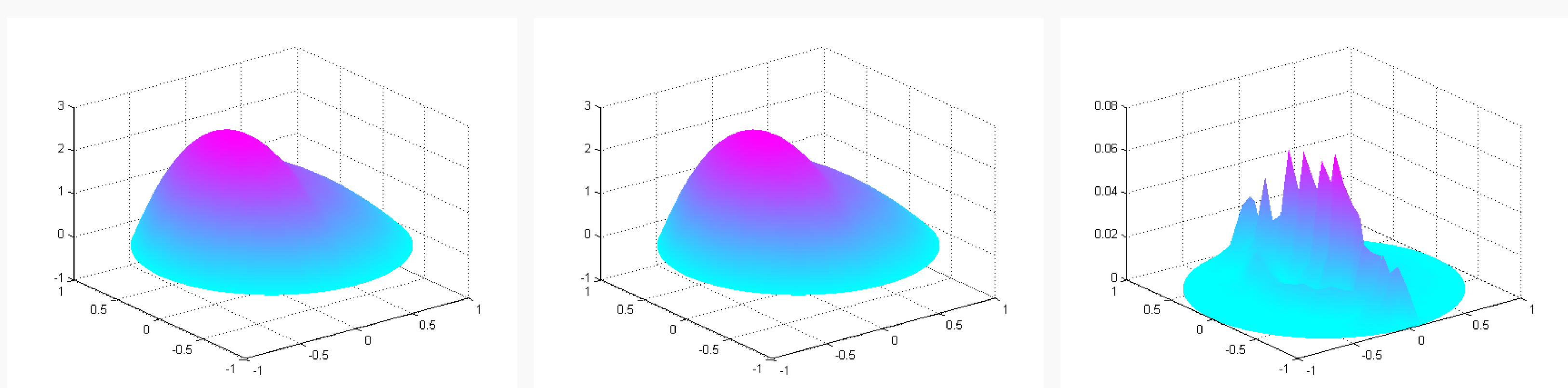
While the higher integrability result is more interesting for the case of systems and matrix-valued k, for applications already the scalar case in dimension $d = 2$ is useful. Here we consider $\Omega := B_1(0)$, $f := 10$, $r = \frac{1}{2}$, $R := 2$, $\gamma := 10^{-4}$, and recover the coefficients k in 2b on the mesh 1 from measurements $\hat{y}$ of the solution u in 3a to the coefficients k in 2a at all triangle midpoints by the MATLAB code below, which needs 97 iterations of FMINCONs active-set algorithm (approx. 1 min) and has errors merely near the jump set.



1 Mesh with 549 finite elements



2 a Original coefficients (left $\frac{1}{2}$, right 2) b Estimated coefficients c Pointwise error



3 a Solution to original coefficients b Solution to estimated coefficients c Pointwise error

```matlab
% geometry and mesh initialization
geom = 'circleg'; bgeom = 'circleb2';
[p,e,t] = initmesh(geom,'Hmax',0.2);
% finite element method
a = 0; f = 10;
i=t(1,:); j=t(2,:); k=t(3,:);
x=(p(1,i)+p(1,j)+p(1,k))/3; y=(p(2,i)+p(2,j)+p(2,k))
   /3;
c=original(x,y)
[K,F,B,UD]=assempde(bgeom,p,e,t,c,a,f);
U=B*(K\F)+UD;
% optimization of the coefficients c resp. C
objfunred = @(CN) objfun(KN,bgeom,p,e,t,a,f,U)
C=pdeprtni(p,t,c);
COPT=fmincon(objfunred,ones(size(C)),[],[],[],[],1/2*
   ones(size(C)),2*ones(size(C)),[],options);
[K,F,B,UD]=assempde(bgeom,p,e,t,pdeintrp(p,t,COPT),a,
   f);
UOPT=B*(K\F)+UD;
% plotting
figure( ... ), pdesurf(p,t, ...), grid on;

function z = objfun(CN,bgeom,p,e,t,a,f,U)
  [K,F,B,UD]=assempde(bgeom,p,e,t,pdeintrp(p,t,CN),a,
    f);
  UN=B*(K\F)+UD;
  CGRAD = pdegrad(p,t,CN); TV = norm(CGRAD,1);
  z = norm(UN-U,2) + TV/1e+04;
end

function c=original(x,y)
  c = 1/2*(x<0) + 2*(x>0);
end
```

References
[1] F. Colonius, K. Kunisch, *Output least squares stability in elliptic systems*, Applies Mathematics and Optimization **19** (1989), 33–63.
[2] D. Cruz-Uribe, K. Moen, S. Rodney, *Regularity results for weak solutions of elliptic PDEs below the natural exponent*, Annali di Matematica Pura ed Applicata **195** (2016), 725–740.
[3] A. Matas, J. Merker, *Model-based estimation of distributed parameters in elliptic systems*, preprint.

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