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CUBATURE FORMULA FOR THE GENERALIZED CHEBYSHEV TYPE POLYNOMIALS

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Abstract: We study cubature formulas to approximate double integrals of generalized Chebyshev-type polynomials of the second type, $\mathbb{U}_{n,r,d}^{(\gamma,M,N)}(U)$, over triangular domain.

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1. Introduction and Motivation

Last few decades have seen a great deal in the field of orthogonal polynomials [1, 13, 22, 23]. Although the main definitions and properties were considered many years ago, the cases of two or more variables of orthogonal polynomials on triangular domains have been studied by few researchers [2, 5, 20, 21]. Proriol introduced the definition of the bivariate orthogonal polynomials on the

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triangle, and the results were summarized by C.F. Dunkl and T. Koornwinder [14, 20].

In addition, recent years have seen a great deal in the field of generalized classical polynomials [3, 4, 8], and their applications [9, 11, 10], the generalized Chebyshev type polynomials of second type are amongst these polynomials.

For $M, N \geq 0$, the generalized Chebyshev-type polynomials of the second type $\left\{\mathscr{U}_n^{(M,N)}(x)\right\}_{n=0}^{\infty}$ (generalized Chebyshev-II) were characterized in [8], these polynomials are orthogonal on the interval $[-1, 1]$ with respect to the generalized weight function

$$\mathrm{W}^{(\gamma,M,N)}(x) = \frac{2}{\pi}(1-x)^{\frac{1}{2}}(1+x)^{\frac{1}{2}} + M\delta(x+1) + N\delta(x-1). \tag{1.1}$$

A closed form for the matrix transformation of the generalized Chebyshev-II polynomial basis into Bernstein polynomial basis, and for Bernstein polynomial basis into generalized Chebyshev-II polynomial basis were provided in [9]. The generalized bivariate Chebyshev-II polynomials $\mathscr{U}_{n,r,d}^{(\gamma,M,N)}(u,v,w)$ are orthogonal to each polynomial of degree $\leq n-1$, with respect to the generalized weight function (1.1). However, for $r \neq s$, $d \neq m$, $\mathscr{U}_{n,r,d}^{(\gamma,M,N)}(u,v,w)$ and $\mathscr{U}_{n,s,m}^{(\gamma,M,N)}(u,v,w)$ are not orthogonal with respect to the weight function.

A construction of bivariate orthogonal polynomials $\mathcal{U}_{n,r}^{(\gamma)}(u,v,w)$, $r = 0, 1, \ldots, n$; $n = 0, 1, 2, \ldots$, with respect to the weight function $u^{\frac{1}{2}}v^{\frac{1}{2}}(1-w)^{\gamma}$, $\gamma > -1$, on a triangular domain were introduced in [6]. They showed that $\mathcal{U}_{n,r}^{(\gamma)}(u,v,w)$ form an orthogonal system. A generalization to the results in [6] were introduced in [7], where for $M, N \geq 0$, the generalized bivariate orthogonal polynomials $\mathbb{U}_{n,r,d}^{(\gamma,M,N)}(u,v,w)$, $d = 0, \ldots, k$; $k = 0, \ldots, n$, $r = 0, 1, \ldots, n$; $n = 0, 1, 2, \ldots$, with respect to the generalized Chebyshev-II weight function (1.1) on triangular domain was given. It was show that $\mathbb{U}_{n,r,d}^{(\gamma,M,N)}(u,v,w)$ form an orthogonal system over the domain T with respect to (1.1). For more details see [2, 6, 7, 12] and references therein.

1.1. Barycentric Coordinates

Consider a triangle T defined by its three vertices $\mathbf{p}_k = (x_k, y_k)$, $k = 1, 2, 3$. For each point $\mathbf{p}$ located inside the triangle, there is a sequence of three numbers $u, v, w \geq 0$ such that $\mathbf{p}$ can be written uniquely as a convex combination of the three vertices, $\mathbf{p} = u\mathbf{p}_1 + v\mathbf{p}_2 + w\mathbf{p}_3$, where $u + v + w = 1$. The three numbers $u = \frac{area(\mathbf{p},\mathbf{p}_2,\mathbf{p}_3)}{area(\mathbf{p}_1,\mathbf{p}_2,\mathbf{p}_3)}$, $v = \frac{area(\mathbf{p}_1,\mathbf{p},\mathbf{p}_3)}{area(\mathbf{p}_1,\mathbf{p}_2,\mathbf{p}_3)}$, $w = \frac{area(\mathbf{p}_1,\mathbf{p}_2,\mathbf{p})}{area(\mathbf{p}_1,\mathbf{p}_2,\mathbf{p}_3)}$ indicate the barycentric "area" coordinates of the point $\mathbf{p}$ with respect to the triangle, where $area(\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3) \neq 0$, which means that $\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3$ are not collinear.

Although there are three coordinates, there are only two degrees of freedom, since $u + v + w = 1$. Thus every point is uniquely defined by any two of the barycentric coordinates. That is, the triangular domain defined as

$$T = \{(u, v, w) : u, v, w \geq 0, u + v + w = 1\}. \tag{1.2}$$

1.2. Bernstein Polynomials

We recall a very concise overview of well-known results on Bernstein polynomials, followed by a brief summary of important properties.

Definition 1. The $n+1$ Bernstein polynomials $B_i^n(x)$ of degree n, $x \in [0, 1], i = 0, 1, \ldots, n$, are defined by:

$$B_i^n(x) = \frac{n!}{i!(n-i)!} x^i (1-x)^{n-i}, \quad i = 0, 1, \ldots, n. \tag{1.3}$$

The Bernstein polynomials have been studied thoroughly and there are a fair amount of literature on these polynomials, they are known for their geometric properties [15, 19], and the Bernstein basis form is known to be optimally stable. They are all non-negative, $B_k^n(x) \geq 0$, $x \in [0, 1]$, has a single unique maximum of $\binom{n}{i} i^i n^{-n} (n-i)^{n-i}$ at $x = \frac{i}{n}$, $i = 0, \ldots, n$, their roots are $x = 0, 1$ with multiplicities, and they form a partition of unity (normalization), satisfy symmetry relation $B_k^n(x) = B_{n-k}^n(1-x)$, and the product of two Bernstein polynomials is also a Bernstein polynomial which is given by $\binom{n+m}{i+j} B_i^n(x) B_j^m(x) = \binom{n}{i}\binom{m}{j} B_{i+j}^{n+m}(x)$.

The Bernstein polynomials of degree n can be defined by combining two Bernstein polynomials of degree $n-1$, where the kth nth-degree Bernstein polynomial can be written by the known recurrence relation as $B_k^n(x) = (1 - x)B_k^{n-1}(x) + xB_{k-1}^{n-1}(x)$, $k = 0, \ldots, n; n \geq 1$ where $B_0^0(x) = 0$ and $B_k^n(x) = 0$ for $k < 0$ or $k > n$. Moreover, it is possible to write each Bernstein polynomials of degree r where $r \leq n$ in terms of Bernstein polynomials of degree n using the following degree elevation [18]:

$$B_k^r(x) = \sum_{i=k}^{n-r+k} \frac{\binom{r}{k}\binom{n-r}{i-k}}{\binom{n}{i}} B_i^n(x), \quad k = 0, 1, \ldots, r. \tag{1.4}$$

For $\zeta = (i, j, k)$ denote triples of non-negative integers such that $|\zeta| = i + j + k$, then the generalized Bernstein polynomials of degree n are defined by the formula $B_\zeta^n(u, v, w) = \binom{n}{\zeta} u^i v^j w^k$, $|\zeta| = n$, where $\binom{n}{\zeta} = \frac{n!}{i!j!k!}$.

The generalized Bernstein polynomials have a number of useful analytical and elegant geometric properties [17]. Note that the generalized Bernstein polynomials are nonnegative over T and form a partition of unity,

$$1 = (u + v + w)^n = \sum_{\substack{0 \le i,j,k \le n \\ i+j+k=n}} \frac{n!}{i!j!k!} u^i v^j w^k. \tag{1.5}$$

These polynomials define the Bernstein basis for the space Π_n, the space of all polynomials of degree at most n. A basis of linearly independent and mutually orthogonal polynomials in the barycentric coordinates (u, v, w) are constructed over T. These polynomials are represented in the following triangular table

$$\begin{array}{lllll}
b_{0,0} & & & & \\
b_{1,0} & b_{1,1} & & & \\
b_{2,0} & b_{2,1} & b_{2,2} & & \\
 & \vdots & & & \\
b_{n,0} & b_{n,1} & b_{n,2} & \dots & b_{n,n}.
\end{array}$$

The kth row of this table contains $k+1$ polynomials. Thus, there are $\frac{(n+1)(n+2)}{2}$ polynomials in a basis of linearly independent polynomials of total degree n. Therefore, the sum (1.5) involves a total of $\frac{(n+1)(n+2)}{2}$ linearly independent polynomials. Thus, with the revolt of computer graphics, Bernstein polynomials on $[0, 1]$ became important in the form of Bézier curves, and the polynomials determined in the Bernstein (Bézier) basis enjoy considerable popularity in Computer Aided Geometric Design applications.

Degree elevation is a common situation in these applications, where polynomials given in the basis of degree n have to be represented in the basis of higher degree.

Any polynomial $p(u, v, w)$ of degree n can be written in the Bernstein form $p(u, v, w) = \sum_{|\zeta|=n} d_\zeta B_\zeta^n(u, v, w)$, with Bézier coefficients d_ζ.

With the use of degree elevation algorithm (1.4) for the Bernstein representation [18], the polynomial $p(u, v, w)$ in (1.4) can be written as

$$p(u, v, w) = \sum_{|\zeta|=n+1} d_\zeta^{(1)} B_\zeta^{n+1}(u, v, w).$$

The new coefficients defined by Hoschek et al. [19] as $d_\zeta^{(1)} = \frac{1}{n+1}(id_{i-1,j,k} + jd_{i,j-1,k} + kd_{i,j,k-1})$ where $|\zeta| = n + 1$. Moreover, the next integration is one of the interesting analytical properties of the Bernstein polynomials $B_\zeta^n(u, v, w)$.

1.3. Integration over Triangular Domains

The integral of a function $f(u, v, w)$ over the triangular domain T defined in (1.2) can be expressed as

$$\iint_T f(u,v,w)dA = A\int_{v=0}^{v=1}\int_{u=0}^{u=1-v} f(u,v,1-v-u)dudv \tag{1.6}$$

where A is the area of T. We can also formulate (1.6) as a double integral over v and w or over u and w, by using $u+v+w=1$. However, for integral of the generalized Bernstein polynomial, we have the following lemma.

Lemma 2. *[16] The Bernstein polynomials $B_\zeta^n(u,v,w), |\zeta| = n$, on T satisfy $\iint_T B_\zeta^n(u,v,w)dA = \frac{\Delta}{\binom{n+2}{2}}$, where Δ is the double the area of T and $\binom{n+2}{2}$ is the dimension of Bernstein polynomials over the triangle.*

This means that the Bernstein polynomials partition the unity with equal integrals over the domain; in other words, they are equally weighted as basis functions.

2. The Generalized Chebyshev-II Polynomials

For $M, N \geq 0$, the generalized Chebyshev-II polynomials $\left\{\mathscr{U}_n^{(M,N)}(x)\right\}_{n=0}^{\infty}$ are orthogonal on the interval $[-1, 1]$ with respect to the weight function (1.1) defined in [20], and been characterized in [8],

$$\mathscr{U}_n^{(M,N)}(x) = \frac{(2n+1)!!}{2^n(n+1)!}U_n(x) + \sum_{k=0}^{n}\lambda_k\frac{(2k+1)!!}{2^k(k+1)!}U_k(x), \tag{2.1}$$

where

$$\lambda_k = \frac{k(k+1)(2k+1)(M+N)}{6} + \frac{(k+2)(k+1)^2k^2(k-1)MN}{9}, \tag{2.2}$$

$U_n(x)$ is the Chebyshev-II polynomial of degree n in x, Szegö [23], and the double factorial of an integer n is given by

$$\begin{aligned}(2n-1)!! &= (2n-1)(2n-3)(2n-5)\dots(3)(1) \quad &&\text{if } n \text{ is odd}\\ n!! &= (n)(n-2)(n-4)\dots(4)(2) &&\text{if } n \text{ is even,}\end{aligned} \tag{2.3}$$

given that $0!! = (-1)!! = 1$.

The next theorem, see [8] for the proof, provides a closed form for generalized Chebyshev-II polynomial $\mathscr{U}_r^{(M,N)}(x)$ of degree r as a linear combination of the Bernstein polynomials $B_i^r(x), i = 0, 1, \dots, r$ of degree r.

Theorem 3. *[8] For $M, N \geq 0$, the generalized Chebyshev-II polynomials $\mathscr{U}_r^{(M,N)}(x)$ of degree r have the following Bernstein representation,*

$$\mathscr{U}_r^{(M,N)}(x) = \frac{(2r+1)!!}{2^r(r+1)!}\sum_{i=0}^{r}(-1)^{r-i}\vartheta_{i,r}B_i^r(x) + \sum_{k=0}^{r}\lambda_k\frac{(2k+1)!!}{2^k(k+1)!}\sum_{i=0}^{k}(-1)^{k-i}\vartheta_{i,k}B_i^k(x)$$

where λ_k defined by (2.2), and

$$\vartheta_{i,r} = \frac{(2r+1)^2}{2^{2r}(2r-2i+1)(2i+1)}\frac{\binom{2r}{r}\binom{2r}{2i}}{\binom{r}{i}}, \ i = 0, 1, \dots, r. \tag{2.4}$$

Now, we have the following corollary which enables us to write Chebyshev-II polynomials of degree r where $r \leq n$ in terms of Bernstein polynomials of degree n.

Corollary 4. *[9] The generalized Chebyshev-II polynomials of degree less than or equal to n, $\mathscr{U}_0^{(M,N)}(x), \dots, \mathscr{U}_n^{(M,N)}(x)$, can be expressed in the Bernstein basis of fixed degree n by the following formula*

$$\mathscr{U}_r^{(M,N)}(x) = \sum_{i=0}^{n}N_{r,i}^nB_i^n(x), \quad r = 0, 1, \dots, n,$$

where

$$N_{r,i}^n = \frac{(2r+1)!!}{2^r(r+1)!}\sum_{k=\max(0,i+r-n)}^{\min(i,r)}\frac{(-1)^{r-k}(2r+1)^2}{2^{2r}(2r-2k+1)(2k+1)}\frac{\binom{n-r}{i-k}\binom{2r}{r}\binom{2r}{2k}}{\binom{n}{i}} + \sum_{k=0}^{r}\lambda_k\frac{(2k+1)!!}{2^k(k+1)!}\sum_{j=\max(0,i+k-n)}^{\min(i,k)}\frac{(-1)^{k-j}(2k+1)^2}{2^{2k}(2k-2j+1)(2j+1)}\frac{\binom{n-k}{i-j}\binom{2k}{k}\binom{2k}{2j}}{\binom{n}{i}}.$$

2.1. Generalized Bivariate Chebyshev-II Polynomials

In this section, we generalize the construction in [6] to formulate a simple closed-form representation of degree-ordered system of generalized orthogonal polynomials $\mathbb{U}_{n,r,d}^{(\gamma,M,N)}(u,v,w)$ on a triangular domain T.

The basic idea in this construction is to make $\mathbb{U}_{n,r,d}^{(\gamma,M,N)}(u,v,w)$ coincide with the univeriate Chebyshev-II polynomial along one edge of T, and to make its variation along each chord parallel to that edge a scaled version of this Chebyshev-II polynomial. The variation of $\mathbb{U}_{n,r,d}^{(\gamma,M,N)}(u,v,w)$ with w can then be arranged so as to ensure its orthogonality on T with every polynomial of degree $< n$, and with other basis polynomials $\mathbb{U}_{n,s,d}^{(\gamma,M,N)}(u,v,w)$ of degree n for $r \neq s$, $d \neq m$.

Now, for $M, N \geq 0$, $\gamma > -1$, $n = 0, 1, 2, \ldots$, $k = 0, \ldots, n$, $r = 0, 1, \ldots, n$ and $d = 0, 1, \ldots, k$, we define the generalized bivariate polynomials

$$\mathbb{U}_{n,r,d}^{(\gamma,M,N)}(u,v,w) = \sum_{i=0}^{r}(-1)^{r-i}\vartheta_{i,r}B_i^r(u,v)\sum_{j=0}^{n-r}(-1)^j\binom{n+r+1}{j}B_j^{n-r}(w,u+v)$$
$$+\sum_{k=0}^{n}\lambda_k\sum_{i=0}^{d}(-1)^{d-i}\vartheta_{i,d}B_i^d(u,v)\sum_{j=0}^{k-d}(-1)^j\binom{k+d+1}{j}B_j^{k-d}(w,u+v), \tag{2.5}$$

where $B_i^r(u,v)$ defined in (1.3), λ_k defined in (2.2), and $\vartheta_{i,r}$ defined in (2.4).

The Bernstein-Bézier form of curves and surfaces exhibits some interesting geometric properties, see [17, 19]. So, for computational purposes, we are interested in finding a closed form of the Bernstein coefficients $a_\zeta^{n,r,d}$, and the recursion relation that allow us to compute the coefficients efficiently.

We write the orthogonal polynomials $\mathbb{U}_{n,r,d}^{(\gamma,M,N)}(u,v,w)$, $r = 0, 1, \ldots, n$, $d = 0, \ldots, k$, and $n = 0, 1, 2, \ldots$ in the following Bernstein-Bézier form,

$$\mathbb{U}_{n,r,d}^{(\gamma,M,N)}(u,v,w) = \sum_{|\zeta|=n} a_\zeta^{n,r,d}B_\zeta^n(u,v,w). \tag{2.6}$$

The following theorem [7] provides a closed explicit form of the Bernstein coefficients $a_\zeta^{n,r,d}$.

Theorem 5. *[7] The Bernstein coefficients $a_\zeta^{n,r,d}$ of equation (2.6) are given explicitly by*

$$a_{ijk}^{n,r,d} = \begin{cases} (-1)^k\binom{n+r+1}{k}\binom{n-r}{k}M_{i,r}^{n-k} + \lambda_k(-1)^j\binom{k+d+1}{j}\binom{k-d}{j}M_{i,d}^{k-j} & \text{if } 0 \leq k \leq n-r, \\ 0 & \text{if } k > n-r, \end{cases}$$

where $M_{i,r}^n$ defined by

$$M_{i,r}^n = \Phi_{i,n}^r + \sum_{k=0}^{r}\lambda_k\Phi_{i,n}^k, \tag{2.7}$$

and

$$\Phi_{i,n}^{r} = \frac{(2r+1)!!}{2^{r}(r+1)!} \sum_{k=\max(0,i+r-n)}^{\min(i,r)} (-1)^{r-k} \frac{\binom{n-r}{i-k}\binom{r+\frac{1}{2}}{k}\binom{r+\frac{1}{2}}{r-k}}{\binom{n}{i}}.$$

Proof. From equation (2.5), it is clear that $\mathbb{U}_{n,r,d}^{(\gamma,M,N)}(u,v,w)$ has degree $\leq n-r$ in the variable w, and thus $a_{ijk}^{n,r} = 0$ for $k > n-r$. For $0 \leq k \leq n-r$, the remaining coefficients are determined by equating (2.5) and (2.6) as follows

$$\begin{aligned}\sum_{i+j=n-k} a_{ijk}^{n,r} B_{ijk}^{n}(u,v,w) = (-1)^{k}\binom{n+r+1}{k} B_{k}^{n-r}(w,u+v) \\ \times \sum_{i=0}^{r} (-1)^{r-i}\vartheta_{i,r} B_{i}^{r}(u,v) \\ + \lambda_{k} \sum_{j=0}^{k-d} (-1)^{j}\binom{k+d+1}{j} B_{j}^{k-d}(w,u+v) \\ \times \sum_{i=0}^{d} (-1)^{d-i}\vartheta_{i,d} B_{i}^{d}(u,v),\end{aligned}$$

where $\gamma > -1$, $B_{i}^{r}(u,v)$, $i = 0,1,\ldots,r$, defined in equation (1.3), and λ_k defined in (2.2). Comparing powers of w on both sides, we have

$$\begin{aligned}\sum_{i=0}^{n-k} a_{ijk}^{n,r} \frac{n!}{i!j!k!} u^{i}v^{j} = (-1)^{k}\binom{n+r+1}{k}\binom{n-r}{k}(u+v)^{n-r-k} \\ \times \sum_{i=0}^{r} (-1)^{r-i}\vartheta_{i,r} B_{i}^{r}(u,v) \\ + \lambda_{k} \sum_{j=0}^{k-d} (-1)^{j}\binom{k+d+1}{j}\binom{k-d}{j}(u+v)^{k-d-j} \\ \times \sum_{i=0}^{d} (-1)^{d-i}\vartheta_{i,d} B_{i}^{d}(u,v).\end{aligned}$$

The left hand side of the last equation can be written in the form

$$\sum_{i=0}^{n-k} a_{ijk}^{n,r} \frac{n!(n-k)!}{i!j!k!(n-k)!} u^{i}v^{j} = \sum_{i=0}^{n-k} a_{ijk}^{n,r} \frac{n!(n-k)!}{i!(n-k-i)!k!(n-k)!} u^{i}v^{j}$$

$$= \sum_{i=0}^{n-k} a_{ijk}^{n,r} \binom{n}{k} B_i^{n-k}(u,v).$$

Now, we have

$$\begin{aligned}\sum_{i=0}^{n-k} a_{ijk}^{n,r} \binom{n}{k} B_i^{n-k}(u,v) = (-1)^k \binom{n+r+1}{k} \binom{n-r}{k} (u+v)^{n-r-k} \\ \times \sum_{i=0}^{r} (-1)^{r-i} \vartheta_{i,r} B_i^r(u,v) \\ + \lambda_k \sum_{j=0}^{k-d} (-1)^j \binom{k+d+1}{j} \binom{k-d}{j} (u+v)^{k-d-j} \\ \times \sum_{i=0}^{d} (-1)^{d-i} \vartheta_{i,d} B_i^d(u,v),\end{aligned}$$

With some binomial simplifications, and using Corollary 4, we get

$$\begin{aligned}\sum_{i=0}^{n-k} a_{ijk}^{n,r} \binom{n}{k} B_i^{n-k}(u,v) = (-1)^k \binom{n+r+1}{k} \binom{n-r}{k} \sum_{i=0}^{r} M_{i,r}^{n-k} B_i^{n-k}(u,v) \\ + \lambda_k \sum_{j=0}^{k-d} (-1)^j \binom{k+d+1}{j} \binom{k-d}{j} \sum_{i=0}^{d} M_{i,d}^{k-j} B_i^{k-j}(u,v),\end{aligned}$$

where $M_{i,r}^{n-k}$ are the coefficients resulting from writing Chebyshev-II polynomial of degree r in the Bernstein basis of degree $n-k$, as defined by expression (2.7). Thus, the required Bernstein-Bézier coefficients are given by

$$a_{ijk}^{n,r,d} = \begin{cases} (-1)^k \binom{n+r+1}{k} \binom{n-r}{k} M_{i,r}^{n-k} + \lambda_k (-1)^j \binom{k+d+1}{j} \binom{k-d}{j} M_{i,d}^{k-j} & \text{if } 0 \le k \le n-r \\ 0 & \text{if } k > n-r. \end{cases}$$

which completes the proof of the theorem. $\square$

3. Cubature Formulas

In this section, we will provide a cubature formula for the generalized Chebyshev type polynomials of second type that will solve multiple integration problems numerically. We provide cubature formula to approximate double integrals over triangular domain.

We defined the polynomial in two variables and the main properties of the Bernstein polynomial over triangular domains, and introduce some definition give some cubature formulas of the form $I(f) = \iint_T f(U)dA$, where $U \in T$, f is defined on the triangular domain $T = \{(u, v, w) : v, u, w \geq 0, u + v + w = 1\}$. For function of one variable, polynomial interpolation and quadrature formula are closely related.

3.1. Cubature Formula over Triangular Domains

In this section, we study interpolatory cubature formula of the form

$$\iint_T f(U)dA = \sum_{|I|=N} C_I f(U_I), \quad C_I \in R, \tag{3.1}$$

we want to choose the appropriate nodes U_I, $|I| = N$ and coefficients C_I, $|I| = N$.

Let $\{B_\zeta^n(U)\}$, $|\zeta| = n$ be the set of generalized Bernstein polynomials over the triangular domain T, where $\zeta = n_1 + n_2 + n_3 = n$, and U be the barycentric coordinates on the triangular domain T. We can interpolate the function f by the generalized Bernstein polynomials and use this interpolation formula to construct a cubature formula as in the following theorem.

Theorem 6. *On the triangular domain T, the generalized bivariate Chebyshevtype polynomials of the second type have the following interpolatory cubature formula*

$$I_n(\mathbb{U}_{n,r,d}^{(\gamma,M,N)}(U)) = \frac{\Delta}{\binom{n+2}{2}} \sum_{|\zeta|=n} a_\zeta^{n,r,d}.$$

Proof. We interpolate $\mathbb{U}_{n,r,d}^{(\gamma,M,N)}(u, v, w)$ using generalized Bernstein polynomials as

$$\mathbb{U}_{n,r,d}^{(\gamma,M,N)}(U) = \sum_{|\zeta|=n} a_\zeta^{n,r,d} B_\zeta^n(U). \tag{3.2}$$

Now, we take the integral to both sides of (3.2) to get

$$\iint_T \mathbb{U}_{n,r,d}^{(\gamma,M,N)}(U)dA = \iint_T \sum_{|\zeta|=n} a_\zeta^{n,r,d} B_\zeta^n(U)dA$$
$$= \sum_{|\zeta|=n} a_\zeta^{n,r,d} \iint_T B_\zeta^n(U)dA. \quad (3.3)$$

Thus,

$$\iint_T \mathbb{U}_{n,r,d}^{(\gamma,M,N)}(U)dA = \frac{\Delta}{\binom{n+2}{2}} \sum_{|\zeta|=n} a_\zeta^{n,r,d} = I_n(\mathbb{U}_{n,r,d}^{(\gamma,M,N)}(U)), \quad (3.4)$$

where Δ is the double the area of T and $\binom{n+2}{2}$ is the dimension of Bernstein polynomials over the triangle, which completes the proof of the interpolatory cubature formula. □

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