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Characterization of the Generalized Gegenbauer Polynomials

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Abstract

We characterize the generalized Gegenbauer polynomials, then we provide a closed form of the the generalized Gegenbauer polynomials using Bernstein basis. We conclude the paper with some results concerning integrals of the generalized Gegenbauer and Bernstein basis.

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1 Introduction

Approximation by polynomials is the oldest and simplest way to represent functions defined over finite domains. It is possible to approximate any arbitrary continuous function by a polynomial and make the error lass than a given accuracy by increasing the degree of the approximating polynomial. On the other side, polynomials can be represented in many different bases such as the monomial power, Gegenbauer, Bernstein, and Hermite basis form. Every type of polynomial basis has its strength and advantages, and sometimes it has disadvantages. Many problems can be solved and many difficulties can be removed by appropriate choice of the basis.

1.1 Properties of Gegenbauer polynomials

Gegenbauer polynomials $C_n^{(\alpha)}(x)$ of degree n generalize Chebyshev polynomials. Chebyshev polynomials of the first kind which denoted by $T_n(x)$ are special cases of Gegenbauer polynomials $C_n^{(\alpha)}(x)$ with $\alpha = -1/2$, and Chebyshev polynomials of the second kind which denoted by $U_n(x)$ are special cases of Gegenbauer polynomials $C_n^{(\alpha)}(x)$ with $\alpha = 1/2$, where Gegenbauer polynomials of degree n in x given by [6],

$$C_n^{(\alpha)}(x) := \frac{n!(2\alpha)!}{(n+2\alpha)!} \sum_{k=0}^{n} \binom{\alpha+n}{n-k} \binom{\alpha+n}{k} \left(\frac{x+1}{2}\right)^{n-k} \left(\frac{x-1}{2}\right)^{k}, \tag{1}$$

are orthogonal, except for a constant factor, on the interval $[-1, 1]$ with respect to the weight function $\mathrm{W}(x) = (1-x)^{\alpha}(1+x)^{\alpha}$, $\alpha > -1$. Although Gegenbauer polynomials defined on $[-1, 1]$, it is more convenient to use $[0, 1]$.

2 Generalized Gegenbauer polynomials

In this section we characterize the generalized Gegenbauer polynomials by generalizing the results of the generalized Chebyshev polynomials in [1, 2], then we provide a closed form for generalized Gegenbauer polynomials $\mathscr{C}_n^{(\alpha,M,N)}(x)$ as a linear combination of the Bernstein polynomials $b_i^r(x)$. We conclude this section with the closed form of the integration of the weighted generalized Gegenbauer with respect to the Bernstein polynomials.

For $\alpha > -1$; $M, N \geq 0$, define $\left\{\mathscr{C}_n^{(\alpha,M,N)}(x)\right\}_{n=0}^{\infty}$ to be the generalized Gegenbauer polynomials as [4]

$$\mathscr{C}_n^{(\alpha,M,N)}(x) = \frac{n!(2\alpha)!}{(n+2\alpha)!} C_n^{(\alpha)}(x) + MQ_n^{(\alpha)}(x) + NR_n^{(\alpha)}(x) + MNS_n^{(\alpha)}(x), \ n = 0, 1, 2, \ldots \tag{2}$$

where $Q_0^{(\alpha)}(x) = R_0^{(\alpha)}(x) = S_0^{(\alpha)}(x) = 0$. For $n = 1, 2, 3, \ldots$

$$Q_n^{(\alpha)}(x) = \frac{n!(2\alpha)!}{(n+2\alpha)!} \frac{(\alpha+2)_{n-1}(2\alpha+2)_{n-1}}{(\alpha+1)_n n!} \times \left[n(n+2\alpha+1)C_n^{(\alpha)}(x) - (\alpha+1)(x-1)DC_n^{(\alpha)}(x)\right], \tag{3}$$

$$R_n^{(\alpha)}(x) = \frac{n!(2\alpha)!}{(n+2\alpha)!} \frac{(\alpha+2)_{n-1}(2\alpha+2)_{n-1}}{(\alpha+1)_n n!} \times \left[n(n+2\alpha+1)C_n^{(\alpha)}(x) - (\alpha+1)(x+1)DC_n^{(\alpha)}(x)\right], \tag{4}$$

$$S_n^{(\alpha)}(x) = \frac{n!(2\alpha)!}{(n+2\alpha)!}\frac{(2\alpha+2)_n(2\alpha+2)_{n-1}}{(\alpha+1)^2 n!(n-1)!} \times [n(n+2\alpha+1)C_n^{(\alpha)}(x) - 2(\alpha+1)xDC_n^{(\alpha)}(x)], \tag{5}$$

where $(x)_n$ defined by $(x)_n = \frac{\Gamma(x+n)}{\Gamma(x)}$, $(x)_0 = 1$.

First remark that the polynomials $\left\{\mathscr{C}_n^{(\alpha,M,N)}(x)\right\}_{n=0}^{\infty}$ satisfy the symmetry relation [5],

$$\mathscr{C}_n^{(\alpha,M,N)}(x) = (-1)^n \mathscr{C}_n^{(\alpha,N,M)}(-x), \quad n = 0,1,2,\dots \tag{6}$$

which implies that for $n = 0,1,2,\dots$ we have $Q_n^{(\alpha)}(x) = (-1)^n R_n^{(\alpha)}(-x)$, and $S_n^{(\alpha)}(x) = (-1)^n S_n^{(\alpha)}(-x)$. Also, from (3) and (4) it follows that

$$Q_n^{(\alpha)}(1) = \frac{(\alpha+2)_{n-1}(2\alpha+2)_n}{(\alpha+1)_n(n-1)!}\frac{n!(2\alpha)!}{(n+2\alpha)!}C_n^{(\alpha)}(1), \quad n = 1,2,3,\dots \tag{7}$$

$$R_n^{(\alpha)}(-1) = \frac{(\alpha+2)_{n-1}(2\alpha+2)_n}{(\alpha+1)_n(n-1)!}\frac{n!(2\alpha)!}{(n+2\alpha)!}C_n^{(\alpha)}(-1), \quad n = 1,2,3,\dots. \tag{8}$$

For $n = 0,1,2,3,\dots$, if we use

$$(x^2-1)D^2C_n^{(\alpha)}(x) = n(n+2\alpha+1)C_n^{(\alpha)}(x) - 2(\alpha+1)xDC_n^{(\alpha)}(x). \tag{9}$$

we easily can write (5) as

$$S_n^{(\alpha)}(x) = \frac{(2\alpha+2)_n(2\alpha+2)_{n-1}}{(\alpha+1)(\alpha+1)n!(n-1)!}(x^2-1)D^2\frac{n!(2\alpha)!}{(n+2\alpha)!}C_n^{(\alpha)}(x), \quad n = 1,2,3,\dots. \tag{10}$$

Note that the representations (3) and (4) and for $n = 1,2,3,\dots$, we have

$$Q_n^{(\alpha)}(x) = \sum_{k=0}^{n} q_k^{(\alpha)}\frac{k!(2\alpha)!}{(k+2\alpha)!}C_k^{(\alpha)}(x) \ \text{ with } \ q_k^{(\alpha)} = \frac{(\alpha+2)_{k-1}(2\alpha+2)_{k-1}}{(\alpha+1)_{k-1}(k-1)!}, \tag{11}$$

and

$$R_n^{(\alpha)}(x) = \sum_{k=0}^{n} r_k^{(\alpha)}\frac{k!(2\alpha)!}{(k+2\alpha)!}C_k^{(\alpha)}(x) \ \text{ with } \ r_k^{(\alpha)} = \frac{(\alpha+2)_{k-1}(2\alpha+2)_{k-1}}{(\alpha+1)_{k-1}(k-1)!}. \tag{12}$$

We also can find from (5) that for $n = 1,2,3,\dots$, we have

$$S_n^{(\alpha)}(x) = \sum_{k=0}^{n} s_k^{(\alpha)}\frac{k!(2\alpha)!}{(k+2\alpha)!}C_k^{(\alpha)}(x) \ \text{ with } \ s_k^{(\alpha)} = \frac{1}{(\alpha+1)^2}\frac{(2\alpha+2)_k(2\alpha+2)_{k-1}}{(k-1)!(k-2)!}. \tag{13}$$

Therefore, for $\alpha > -1$; $M, N \geq 0$, the generalized Gegenbauer polynomials $\left\{\mathscr{C}_n^{(\alpha,M,N)}(x)\right\}_{n=0}^{\infty}$ are orthogonal on the interval $[-1, 1]$ with respect to the weight function [5]

$$\frac{\Gamma(2\alpha+2)}{2^{2\alpha+1}\Gamma(\alpha+1)^2}(1-x)^{\alpha}(1+x)^{\alpha} + M\delta(x+1) + N\delta(x-1), \qquad (14)$$

and can be written as

$$\mathscr{C}_n^{(\alpha,M,N)}(x) = \frac{n!(2\alpha)!}{(n+2\alpha)!}C_n^{(\alpha)}(x) + \sum_{k=0}^{n} \lambda_k \frac{k!(2\alpha)!}{(k+2\alpha)!}C_k^{(\alpha)}(x), \qquad (15)$$

where

$$\lambda_k = Mq_k^{(\alpha)} + Nr_k^{(\alpha)} + MNs_k^{(\alpha)}. \qquad (16)$$

2.1 Construction of generalized Gegenbauer polynomials using Bernstein basis

Bernstein polynomials have been studied thoroughly and there are a fair amount of literature on theses polynomials.

Definition 2.1. *The $n+1$ Bernstein polynomials $b_i^n(x)$ of degree n, $x \in [0, 1], i = 0, 1, \ldots, n$, are defined by:*

$$b_i^n(x) = \begin{cases} \binom{n}{i} x^i (1-x)^{n-i} & i = 0, 1, \ldots, n \\ 0 & else \end{cases}, \qquad (17)$$

where the binomial coefficients $\binom{n}{i} = \frac{n!}{i!(n-i)!}$, $\quad i = 0, \ldots, n$.

Bernstein polynomials are known [3] for their analytic and geometric properties, and their basis is known to be optimally stable. They are all non-negative, $b_i^n(x) \geq 0$, $x \in [0, 1]$, satisfy symmetry relation $b_i^n(x) = b_{n-i}^n(1-x)$, has a single unique maximum of $\binom{n}{i} i^i n^{-n}(n-i)^{n-i}$ at $x = \frac{i}{n}$, $i = 0, \ldots, n$, their roots are $x = 0, 1$ with multiplicities, and they form a partition of unity (normalization). The kth nth-degree Bernstein polynomial can be defined by two Bernstein polynomials of degree $n-1$ as

$$b_k^n(x) = (1-x)b_k^{n-1}(x) + xb_{k-1}^{n-1}(x), \quad k = 0, \ldots, n; n \geq 1$$

where $b_0^0(x) = 0$ and $b_k^n(x) = 0$ for $k < 0$ or $k > n$. Moreover, the product of two Bernstein polynomials is also a Bernstein polynomial and given by

$$\binom{n+m}{i+j} b_i^n(x) b_j^m(x) = \binom{n}{i}\binom{m}{j} b_{i+j}^{n+m}(x).$$

These properties and others [3] make the Bernstein polynomials important for the development of Bézier curves and surfaces in Computer Aided Geometric Design. The Bernstein polynomials are actually the standard basis for the Bézier representations of curves and surfaces in Computer Aided Geometric Design. However, the Bernstein polynomials are not orthogonal and could not be used effectively in the least-squares approximation [3], and thus the calculations performed in obtaining the least-square approximation polynomial of degree n do not reduce the calculations to obtain the least-squares approximation polynomial of degree $n+1$.

Since then a theory of approximation has been developed and many approximation methods have been introduced and analyzed. The method of least squares approximation accompanied by orthogonal polynomials is one of theses approximation methods.

Choosing $\{\varphi_0(x), \varphi_1(x), \dots, \varphi_n(x)\}$ to be orthogonal greatly simplifies the least-squares approximation problem. The matrix of the normal equations is diagonalized, which simplifies calculations and gives a compact form for the Least-Squares Polynomial coefficients. See [3] for more details on Bernstein polynomials and the least squares approximations.

To write a generalized Gegenbauer polynomial $\mathscr{C}_r^{(\alpha,M,N)}(x)$ of degree r as a linear combination of the Bernstein polynomial basis $b_i^r(x), i = 0, 1, \dots, r$ of degree r in explicit form, we begin with substituting (17) into (15) to get

$$\mathscr{C}_r^{(\alpha,M,N)}(x) = \frac{1}{\binom{r+2\alpha}{r}} \sum_{i=0}^{r} \frac{\binom{r+\alpha}{r-i}\binom{r+\alpha}{i}}{\binom{r}{r-i}} b_{r-i}^r(x) + \sum_{k=0}^{r} \lambda_k \frac{k!(2\alpha)!}{(k+2\alpha)!} \sum_{j=0}^{k} \frac{\binom{k+\alpha}{k-j}\binom{k+\alpha}{j}}{\binom{k}{k-j}} b_{k-j}^k(x).$$

By using the Bernstein polynomials symmetry relation, rearranging the terms, and defining

$$\eta_{i,r}^{(\alpha)} = \frac{\binom{r+\alpha}{i}\binom{r+\alpha}{r-i}}{\binom{r}{i}}, i = 0, 1, \dots, r,$$

we show that the generalized Gegenbauer polynomial $\mathscr{C}_r^{(\alpha,M,N)}(x)$ of degree r can be written in the Bernstein basis form as

$$\mathscr{C}_r^{(\alpha,M,N)}(x) = \frac{1}{\binom{r+2\alpha}{r}} \sum_{i=0}^{r} (-1)^{r-i} \eta_{i,r}^{(\alpha)} b_i^r(x) + \sum_{k=0}^{r} \lambda_k \frac{k!(2\alpha)!}{(k+2\alpha)!} \sum_{j=0}^{k} (-1)^{k-j} \eta_{j,k}^{(\alpha)} b_j^k(x).$$

In addition, using simple combinatorial identities to simplify $\eta_{i-1,r}^{(\alpha)}$, we have

$$\eta_{i-1,r}^{(\alpha)} = \frac{(i+\alpha)}{(r+\alpha-i+1)} \eta_{i,r}^{(\alpha)}, \quad i = 1, \dots, r.$$

Thus, the next theorem provides a closed form for generalized Gegenbauer polynomials $\mathscr{C}_r^{(\alpha,M,N)}(x)$ of degree r as a linear combination of the Bernstein polynomials $b_i^r(x), i = 0, 1, \dots, r$.

Theorem 2.2. *Let $\alpha > -1$; $M, N \geq 0$, the generalized Gegenbauer polynomials $\mathscr{C}_r^{(\alpha,M,N)}(x)$ of degree r have the following Bernstein representation:*

$$\mathscr{C}_r^{(\alpha,M,N)}(x) = \frac{r!(2\alpha)!}{(r+2\alpha)!}\sum_{i=0}^{r}(-1)^{r-i}\eta_{i,r}^{(\alpha)}b_i^r(x)+\sum_{k=0}^{r}\frac{k!(2\alpha)!\lambda_k}{(k+2\alpha)!}\sum_{i=0}^{k}(-1)^{k-i}\eta_{i,k}^{(\alpha)}b_i^k(x) \tag{18}$$

where $\lambda_k = Mq_k^{(\alpha)} + Nr_k^{(\alpha)} + MNs_k^{(\alpha)}$, $\eta_{0,r}^{(\alpha)} = \binom{r+\alpha}{r}$, and

$$\eta_{i,r}^{(\alpha)} = \frac{\binom{r+\alpha}{i}\binom{r+\alpha}{r-i}}{\binom{r}{i}}, i = 0, 1, \ldots, r.$$

The coefficients $\eta_{i,r}^{(\alpha)}$ satisfy the recurrence relation

$$\eta_{i,r}^{(\alpha)} = \frac{(r+\alpha-i+1)}{(i+\alpha)}\eta_{i-1,r}^{(\alpha)}, \quad i = 1, \ldots, r. \tag{19}$$

We conclude with an interesting integration property of the weighted generalized Gegenbauer polynomials with the Bernstein polynomials. First, recall that the Bernstein polynomials can be differentiated and integrated easily as

$$\frac{d}{dx}b_k^n(x) = n[b_{k-1}^{n-1}(x) - b_k^{n-1}(x)], \quad n \geq 1,$$

and

$$\int_0^1 b_k^n(x)dx = \frac{1}{n+1}, \quad k = 0, 1, \ldots, n.$$

Moreover, we introduce the Eulerian integral of the first kind which is often called Beta integral.

Definition 2.3. *The Eulerian integral of the first kind is a function of two complex variables defined by*

$$\int_0^1 u^{x-1}(1-u)^{y-1}du = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}, \qquad \Re(x), \Re(y) > 0.$$

We observe that the beta integral is symmetric, a change of variables by $t = 1-u$ clearly illustrates this. The following theorem shows the integration of the weighted generalized Gegenbauer with respect to the Bernstein polynomials.

Theorem 2.4. *Let $b_r^n(x)$ be the Bernstein polynomial of degree n and $\mathscr{C}_i^{(\alpha,M,N)}(x)$ be the generalized Gegenbauer polynomial of degree i, then for $i, r = 0, 1, \ldots, n$ we have*

$$\int_0^1 (1-x)^\alpha x^\alpha b_r^n(x)\mathscr{C}_i^{(\alpha,M,N)}(x)dx = \Psi_{r,\alpha}^{i,n} + \sum_{d=0}^{i}\lambda_d\Psi_{r,\alpha}^{d,n}, \tag{20}$$

where λ_d *defined in* (16),

$$\Psi_{r,\alpha}^{d,n} = \binom{n}{r} \frac{d!(2\alpha)!}{(d+2\alpha)!} \times \sum_{j=0}^{d} (-1)^{d-j} \binom{d+\alpha}{j} \binom{d+\alpha}{d-j} \frac{\Gamma(\alpha+r+j+1)\Gamma(n+d+\alpha-r-j+1)}{\Gamma(n+d+2\alpha+2)},$$

and $\Gamma(x)$ *is the Gamma function.*

Proof. By using (18), the integral $I = \int_0^1 (1-x)^\alpha x^\alpha b_r^n(x) \mathscr{C}_i^{(\alpha,M,N)}(x)dx$, can be simplified to

$$\begin{aligned}
I &= \int_0^1 (1-x)^\alpha x^\alpha \binom{n}{r} x^r (1-x)^{n-r} \frac{i!(2\alpha)!}{(i+2\alpha)!} \sum_{k=0}^{i} (-1)^{i-k} \frac{\binom{i+\alpha}{k}\binom{i+\alpha}{i-k}}{\binom{i}{k}} b_k^i(x) \\
&+ \sum_{d=0}^{i} \lambda_d \frac{d!(2\alpha)!}{(d+2\alpha)!} \int_0^1 (1-x)^\alpha x^\alpha \binom{n}{r} x^r (1-x)^{n-r} \sum_{j=0}^{d} (-1)^{d-j} \frac{\binom{d+\alpha}{j}\binom{d+\alpha}{d-j}}{\binom{d}{j}} b_j^d(x) dx \\
&= \binom{n}{r} \frac{i!(2\alpha)!}{(i+2\alpha)!} \sum_{k=0}^{i} (-1)^{i-k} \binom{i+\alpha}{k} \binom{i+\alpha}{i-k} \int_0^1 x^{\alpha+r+k} (1-x)^{n+i+\alpha-r-k} dx \\
&+ \sum_{d=0}^{i} \lambda_d \binom{n}{r} \frac{d!(2\alpha)!}{(d+2\alpha)!} \sum_{j=0}^{d} (-1)^{d-j} \binom{d+\alpha}{j} \binom{d+\alpha}{d-j} \int_0^1 x^{\alpha+r+j} (1-x)^{n+d+\alpha-r-j} dx.
\end{aligned}$$

The integrals in the last equation are the Eulerian integrals with $x_1 = \alpha+r+k+1$, $y_1 = n+i+\alpha-r-k+1$, $x_2 = \alpha+r+j+1$, and $y_2 = n+d+\alpha-r-j+1$. $\square$

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