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Constrained Ultraspherical-Weighted Orthogonal Polynomials on Triangle

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Abstract

We construct Ultraspherical-weighted orthogonal polynomials $\mathscr{C}_{n,r}^{(\lambda,\gamma)}(u,v,w)$, $\lambda > -\frac{1}{2}$, $\gamma > -1$, on the triangular domain T, where $2\lambda + \gamma = 1$. We show $\mathscr{C}_{n,r}^{(\lambda,\gamma)}(u,v,w)$, $r = 0, 1, \ldots, n$; $n \geq 0$ form an orthogonal system over the triangular domain T with respect to the Ultraspherical weight function.

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1 Introduction

Recent years have seen a great deal in the field of orthogonal polynomials, the Ultraspherical orthogonal polynomials are Amongst these polynomials [1, 2, 10, 13, 20]. Although the main definitions and properties were considered many years ago, the cases of two or more variables of orthogonal polynomials on triangular domains have been studied by few researchers [11, 12, 19]. Proriol [15] introduced the definition of the bivariate orthogonal polynomials on the triangle, and the results were summarized by C.F. Dunkl and T. Koornwinder [5, 11].

Orthogonal polynomials with Ultraspherical weight function $\mathrm{W}^{(\lambda,\gamma)}(u,v,w) = u^{\lambda-\frac{1}{2}}v^{\lambda-\frac{1}{2}}(1-w)^{\gamma}, \lambda > \frac{-1}{2}, \gamma > -1$ on triangular domain T are defined in many articles and textbooks, for instance [3, 10]. These polynomials $C_{n,r}^{(\lambda,\gamma)}(u,v,w)$, are orthogonal to each polynomial of degree less than or equal to $n-1$, with respect to the defined weight function $\mathrm{W}^{(\lambda,\gamma)}(u,v,w)$ on T. However, for $r \neq s$, $C_{n,r}^{(\lambda,\gamma)}(u,v,w)$ and $C_{n,s}^{(\lambda,\gamma)}(u,v,w)$ are not orthogonal with respect to the weight function $\mathrm{W}^{(\lambda,\gamma)}(u,v,w)$ on T.

S. Waldron start the work of a generalized beta integral and the limit of the Bernstein-Durrmeyer operator with Jacobi weights. Also, he computed orthogonal polynomials on a triangle by degree raising. Farouki [7] defined the orthogonal polynomials with respect to the weight function $\mathrm{W}(u,v,w) = 1$ on a triangular domain T. These polynomials $P_{n,r}(u,v,w)$ defined in [7], are orthogonal to each polynomial of degree $\leq n-1$ and also orthogonal to each polynomial $P_{n,s}(u,v,w)$, $r \neq s$.

In this paper, we construct orthogonal polynomials $\mathscr{C}_{n,r}^{(\lambda,\gamma)}(u,v,w)$, with respect to the Ultraspherical weight function $\mathrm{W}^{(\lambda,\gamma)}(u,v,w) = u^{\lambda-\frac{1}{2}}v^{\lambda-\frac{1}{2}}(1-w)^{\gamma}, \lambda > \frac{-1}{2}, \gamma > -1$, on triangular domain T, such that $2\lambda+\gamma = 1$. These Ultraspherical-weighted orthogonal polynomials are given in terms of Bernstein basis, so many geometric properties of the Bernstein polynomial basis are preserve. We show that these bivariate polynomials $\mathscr{C}_{n,r}^{(\lambda,\gamma)}(u,v,w)$, $r = 0,1,\ldots,n$, and $n = 0,1,2,\ldots$, form an orthogonal system over the triangular domain T with respect to the weight function $\mathrm{W}^{(\lambda,\gamma)}(u,v,w) = u^{\lambda-\frac{1}{2}}v^{\lambda-\frac{1}{2}}(1-w)^{\gamma}, \lambda > \frac{-1}{2}, \gamma > -1$, where $2\lambda+\gamma = 1$.

On the triangular domain T, we proved that these polynomials $\mathscr{C}_{n,r}^{(\lambda,\gamma)}(u,v,w) \in \mathfrak{L}_{\mathfrak{n}}$, $n \geq 1$, $r = 0,1,\ldots,n$, and for $r \neq s$, $\mathscr{C}_{n,r}^{(\lambda,\gamma)}(u,v,w) \perp \mathscr{C}_{n,s}^{(\lambda,\gamma)}(u,v,w)$.

P.K. Suetin [19] constructed bivariate orthogonal polynomials on the square. He considered the tensor product of the set of orthogonal polynomials over the domain $G = \{(x,y) : -1 \leq x \leq 1, -1 \leq y \leq 1\}$.

Let $\{C_n^{(\lambda_1)}(x)\}$, $\{Q_m^{(\lambda_2)}(y)\}$ be the Ultraspherical polynomials over $[-1,1]$ with respect to the weight functions $\mathrm{W}_1(x) = (1-x^2)^{\lambda_1-\frac{1}{2}}$, and $\mathrm{W}_2(y) = (1-y^2)^{\lambda_2-\frac{1}{2}}$ respectively. P.K. Suetin [19], defined the bivariate polynomials $\{R_{nm}(x,y)\}$ on G formed by the tensor products of the Ultraspherical polynomials by

$$R_{nm}(x,y) := C_{n-m}^{(\lambda_1)}(x)Q_m^{(\lambda_2)}(y), n = 0,1,2,\ldots, m = 0,1,\ldots,n.$$

Then $\{R_{nm}(x,y)\}$ are orthogonal on the square G with respect to the weight function $\mathrm{W}(x,y) = \mathrm{W}_1^{(\lambda_1)}(x)\mathrm{W}_2^{(\lambda_2)}(y)$. However, The construction of orthogonal polynomials over a triangular domain is not straightforward like the tensor product over the square.

2 Barycentric, and Bernstein Polynomials

Consider a base triangle in the plane with the vertices $\mathbf{p}_k = (x_k, y_k)$, $k = 1, 2, 3$. Then every point $\mathbf{p}$ inside the triangle

$$T = \{(u, v, w) : u, v, w \geq 0, u + v + w = 1\},$$

can be written using the barycentric coordinates (u, v, w), as $\mathbf{p}=u\mathbf{p}_1 + v\mathbf{p}_2 + w\mathbf{p}_3$. The barycentric coordinates are given in the following ratios:

$$u = \frac{area(\mathbf{p}, \mathbf{p}_2, \mathbf{p}_3)}{area(\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3)}, \quad v = \frac{area(\mathbf{p}_1, \mathbf{p}, \mathbf{p}_3)}{area(\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3)}, \quad w = \frac{area(\mathbf{p}_1, \mathbf{p}_2, \mathbf{p})}{area(\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3)}.$$

where $area(\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3) \neq 0$, which means that $\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3$ are not collinear.

Let the notation $\zeta = (i, j, k)$ denote triples of nonnegative integers, where $|\zeta| = i + j + k$. The generalized Bernstein polynomials of degree n on the triangular domain T are defined by the formula

$$b^n_\zeta(u, v, w) = \binom{n}{\zeta} u^i v^j w^k, \quad |\zeta| = n, \quad \text{where } \binom{n}{\zeta} = \frac{n!}{i!j!k!}.$$

Note that the generalized Bernstein polynomials are nonnegative over T, and form a partition of unity,

$$1 = (u + v + w)^n = \sum_{\substack{0 \leq i,j,k \leq n \\ i+j+k=n}} \frac{n!}{i!j!k!} u^i v^j w^k.$$

These polynomials define the Bernstein basis for the space Π_n over the triangular domain T, where the kth row contains $k + 1$ polynomials. Thus, for a basis of linearly independent polynomials of total degree n, there are a total of $(1/2)(n + 1)(n + 2)$ linearly independent polynomials.

Any polynomial $p(u, v, w)$ of degree n can be written in the Bernstein form as

$$p(u, v, w) = \sum_{|\zeta|=n} d_\zeta b^n_\zeta(u, v, w), \tag{1}$$

with Bézier coefficients d_ζ. We can also use the degree elevation algorithm for the Bernstein representation (1). This is obtained by multiplying both sides by $1 = u + v + w$, and writing

$$p(u, v, w) = \sum_{|\zeta|=n+1} d^{(1)}_\zeta b^{n+1}_\zeta(u, v, w),$$

the new coefficients $d^{(1)}_\zeta$ are defined by, see [6, 9] ,

$$d^{(1)}_{i,j,k} = \frac{1}{n + 1}(id_{i-1,j,k} + jd_{i,j-1,k} + kd_{i,j,k-1}), \quad i + j + k = n + 1.$$

The Bernstein polynomials $b^n_\zeta(u,v,w), |\zeta| = n$, on T satisfy, see [7],

$$\iint_T b^n_\zeta(u,v,w)dA = \frac{\Delta}{(n+1)(n+2)},$$

where Δ is double the area of T.

Let $p(u,v,w)$ and $q(u,v,w)$ be two bivariate polynomials over T, then we define their inner product over T by

$$\langle p,q\rangle = \frac{1}{\Delta}\iint_T pq dA.$$

We say that p and q are orthogonal if $\langle p,q\rangle = 0$.

For $m \geq 1$, we define $\mathfrak{L}_\mathfrak{m} = \{p \in \Pi_m : p \perp \Pi_{m-1}\}$ to be the space of polynomials of degree m that are orthogonal to all polynomials of degree $< m$ over a triangular domain T, and Π_n is the space of all polynomials of degree n over the triangular domain T.

Let $f(u,v,w)$ be an integrable function over T and consider the operator

$$S_n(f) = (n+1)(n+2)\sum_{|\zeta|=n} \left\langle f, b^n_\zeta\right\rangle b^n_\zeta.$$

For $n \geq m$,

$$\lambda_{m,n} = \frac{(n+2)!n!}{(n+m+2)!(n-m)!}$$

is an eigenvalue of the operator S_n and $\mathfrak{L}_\mathfrak{m}$ is the corresponding eigenspace, see [4] for proof and more details. The following lemmas will be needed in the proof of the main results, see [7, 14] for the proofs and more details.

Lemma 2.1. *(See [7]). Let* $p = \sum_{|\zeta|=n} c_\zeta b^n_\zeta \in \mathfrak{L}_\mathfrak{m}$ *and let* $q = \sum_{|\zeta|=n} d_\zeta b^n_\zeta \in \Pi_n$ *with* $m \leq n$*. Then,*

$$\langle p,q\rangle = \frac{(n!)^2}{(n+m+2)!(n-m)!}\sum_{|\zeta|=n} c_\zeta d_\zeta.$$

Lemma 2.2. *(See [14]). Let* $p \in \sum_{|\zeta|=n} c_\zeta b^n_\zeta \in \mathfrak{L}_\mathfrak{n}$*. Then,*

$$p \in \mathfrak{L}_\mathfrak{n} \iff \sum_{|\zeta|=n} c_\zeta d_\zeta = 0 \;\; \forall q = \sum_{|\zeta|=n} d_\zeta b^n_\zeta \in \Pi_{n-1}. \tag{2}$$

3 Ultraspherical Polynomials

The Ultraspherical polynomials $C^{(\lambda)}_n(x)$ of degree n are the orthogonal polynomials, except for a constant factor, on $[-1,1]$ with respect to the weight function

$$\mathrm{W}(x) = (1-x^2)^{\lambda-\frac{1}{2}}, \lambda > -\frac{1}{2}.$$

In this paper, it is appropriate to take $x \in [0,1]$ for both Bernstein and Ultraspherical polynomials.

The following lemmas, See A. Rababah [17], will be needed in the construction of the orthogonal bivariate polynomials and the proof of the main results. For more details and the proofs, see [17]. Although the Pochhammer symbol is more appropriate, the combinatorial notation will be used, Szegö [20], since it is more compact and readable formulas.

Lemma 3.1. *The Ultraspherical polynomials $C_r^{(\lambda)}(x)$ have the Bernstein representation:*

$$C_r^{(\lambda)}(x) = \frac{(\lambda+\frac{1}{2})_n}{(2\lambda)_n} \sum_{i=0}^{r} (-1)^{r-i} \frac{\binom{r+\lambda-\frac{1}{2}}{i}\binom{r+\lambda-\frac{1}{2}}{r-i}}{\binom{r}{i}} b_i^r(x), \; r = 0, 1, \ldots . \tag{3}$$

Lemma 3.2. *The Ultraspherical polynomials $C_0^{(\lambda)}(x), \ldots, C_n^{(\lambda)}(x)$ of degree $\leq n$ can be expressed in the Bernstein basis of fixed degree n by the following formula*

$$C_r^{(\lambda)}(x) = \sum_{i=0}^{n} \mu_{i,r}^n b_i^n(x), \quad r = 0, 1, \ldots, n$$

where

$$\mu_{i,r}^n = \frac{(\lambda+\frac{1}{2})_n}{(2\lambda)_n} \binom{n}{i}^{-1} \sum_{k=\max(0,i+r-n)}^{\min(i,r)} (-1)^{r-k} \binom{n-r}{i-k} \binom{r+\lambda-\frac{1}{2}}{k} \binom{r+\lambda-\frac{1}{2}}{r-k} \tag{4}$$

In addition, the following combinatorial identity, Lemma 3.3 [18], can be used for the main results simplifications.

Lemma 3.3. *For an integer n, we have the following combinatorial identity*

$$\binom{n-\frac{1}{2}}{n-k}\binom{n-\frac{1}{2}}{k} = \frac{1}{2^{2n}} \binom{2n}{n}\binom{2n}{2k}.$$

In the following lemma, let

$$q_{n,r}(w) = \sum_{j=0}^{n-r} (-1)^j \binom{n+r+1}{j} b_j^{n-r}(w). \tag{5}$$

Lemma 3.4. *(See [7]). For $r = 0, \ldots, n$ and $i = 0, \ldots, n-r-1$, $q_{n,r}(w)$ is orthogonal to $(1-w)^{2r+i+1}$ on $[0,1]$, and hence, for every polynomial $p(w)$ of degree $\leq n-r-1$,*

$$\int_0^1 q_{n,r}(w) p(w) (1-w)^{2r+1} dw = 0.$$

4 Ultraspherical-Weighted Polynomials

Analogous to [7], a simple closed-form representation of degree-ordered system of orthogonal polynomials is constructed on a triangular domain T. Since the Bernstein polynomials are stable [8], it is convenient to write these polynomials in Bernstein form.

For $n = 0, 1, 2, \dots$ and $r = 0, 1, \dots, n$ we define the bivariate polynomials

$$\mathscr{C}_{n,r}^{(\lambda,\gamma)}(u,v,w) = \sum_{i=0}^{r} c(i,\lambda) b_i^r(u,v) \sum_{j=0}^{n-r} (-1)^j \binom{n+r+1}{j} b_j^{n-r}(w, u+v), \quad (6)$$

where $\lambda > -\frac{1}{2}, \gamma > -1$, $2\lambda + \gamma = 1$, $b_i^r(u,v) = \binom{r}{i} u^i v^{r-i}$, $\; i = 0, 1, \dots, r$, and

$$c(i,\lambda) = (-1)^{r-i} \frac{\binom{r+\lambda-\frac{1}{2}}{i}\binom{r+\lambda-\frac{1}{2}}{r-i}}{\binom{r}{i}}, \quad i = 0, 1, \dots, r. \quad (7)$$

In this section, we show that the polynomials $\mathscr{C}_{n,r}^{(\lambda,\gamma)}(u,v,w) \in \mathfrak{L}_n$, $r = 0, 1, \dots, n$; $n \geq 1$, and for $r \neq s$, $\mathscr{C}_{n,r}^{(\lambda,\gamma)} \perp \mathscr{C}_{n,s}^{(\lambda,\gamma)}$. By choosing $\mathscr{C}_{0,0}^{(\lambda,\gamma)} = 1$, the polynomials $\mathscr{C}_{n,r}^{(\lambda,\gamma)}(u,v,w)$ for $0 \leq r \leq n$ and $n \geq 0$ form a degree-ordered orthogonal sequence over T.

We first rewrite these polynomials in the Ultraspherical polynomials form:

$$\begin{aligned}\mathscr{C}_{n,r}^{(\lambda,\gamma)}(u,v,w) &= \sum_{i=0}^{r} c(i,\lambda) b_i^r(u,v) \sum_{j=0}^{n-r} (-1)^j \binom{n+r+1}{j} b_j^{n-r}(w, u+v) \\ &= \sum_{i=0}^{r} c(i,\lambda) \frac{b_i^r(u,v)}{(u+v)^r} (1-w)^r \sum_{j=0}^{n-r} (-1)^j \binom{n+r+1}{j} b_j^{n-r}(w, 1-w).\end{aligned}$$

Since $b_i^r(u,v) = (u+v)^r b_i^r(\frac{u}{1-w})$, and using Lemma 3.1 we get

$$\mathscr{C}_{n,r}^{(\lambda,\gamma)}(u,v,w) = \frac{(\lambda+\frac{1}{2})_n}{(2\lambda)_n} C_r^{(\lambda)}(\frac{u}{1-w})(1-w)^r q_{n,r}(w), \quad r = 0, \dots, n, \quad (8)$$

where $C_r^{(\lambda)}(t)$ is the univariate Ultraspherical polynomial of degree r and $q_{n,r}(w)$ is defined in equation (5).

For simplicity, since we are dealing with orthogonality, and the Ultraspherical polynomials $C_n(x)$ of degree n are the orthogonal except for a constant factor, we rewrite (8) as

$$\mathscr{C}_{n,r}^{(\lambda,\gamma)}(u,v,w) = C_r^{(\lambda)}(\frac{u}{1-w})(1-w)^r q_{n,r}(w), \quad r = 0, \dots, n. \quad (9)$$

First, we show that the polynomials $\mathscr{C}_{n,r}^{(\lambda,\gamma)}(u,v,w)$, $r = 0, \dots, n$, are orthogonal to all polynomials of degree less than n over the triangular domain T.

Theorem 4.1. *For each* $n = 1, 2, \ldots, r = 0, 1, \ldots, n$, *and the weight function* $\mathrm{W}^{(\lambda,\gamma)}(u,v,w) = u^{\lambda-\frac{1}{2}}v^{\lambda-\frac{1}{2}}(1-w)^{\gamma}$ *such that* $\lambda > -\frac{1}{2}, \gamma > -1, 2\lambda+\gamma = 1$, $\mathscr{C}_{n,r}^{(\lambda,\gamma)}(u,v,w) \in \mathfrak{L}_n$.

Proof. For each $m = 0, \ldots, n-1$, and $s = 0, \ldots, m$ we construct the set of bivariate polynomials

$$Q_{s,m}^{(\lambda)}(u,v,w) = C_s^{(\lambda)}\left(\frac{u}{1-w}\right)(1-w)^m w^{n-m-1}, \; m = 0, \ldots, n-1, s = 0, \ldots, m. \tag{10}$$

The span of these polynomials includes the set of Bernstein polynomials

$$b_j^m(\frac{u}{1-w})(1-w)^m w^{n-m-1} = b_j^m(u,v)w^{n-m-1} \quad m = 0, \ldots, n-1, j = 0, \ldots, m,$$

which span Π_{n-1}. Thus, it is sufficient to show that for each $m = 0, \ldots, n-1$, $s = 0, \ldots, m$, we have

$$I := \iint\limits_T \mathscr{C}_{n,r}^{(\lambda,\gamma)}(u,v,w)Q_{s,m}^{(\lambda)}(u,v,w)\mathrm{W}^{(\lambda,\gamma)}(u,v,w)dA = 0. \tag{11}$$

This is simplified to

$$= \Delta \int_0^1 \int_0^{1-w} C_r^{(\lambda)}(\frac{u}{1-w})q_{n,r}(w)C_s^{(\lambda)}(\frac{u}{1-w})w^{n-m-1}u^{\lambda-\frac{1}{2}}v^{\lambda-\frac{1}{2}}(1-w)^{\gamma+r+m}dudw. \tag{12}$$

By making the substitution $t = \frac{u}{1-w}$, we get

$$I = \Delta \int_0^1 \int_0^1 C_r^{(\lambda)}(t)q_{n,r}(w)C_s^{(\lambda)}(t)(1-w)^{2\lambda+\gamma+r+m}w^{n-m-1}t^{\lambda-\frac{1}{2}}(1-t)^{\lambda-\frac{1}{2}}\,dtdw$$

$$= \Delta \int_0^1 C_r^{(\lambda)}(t)C_s^{(\lambda)}(t)t^{\lambda-\frac{1}{2}}(1-t)^{\lambda-\frac{1}{2}}\,dt \int_0^1 q_{n,r}(w)(1-w)^{2\lambda+\gamma+r+m}w^{n-m-1}dw.$$

If $m < r$, then we have $s < r$, and the first integral is zero by the orthogonality property of the Ultraspherical polynomials. If $r \leq m \leq n-1$, we have by Lemma 3.4 the second integral equals zero, Thus the theorem follows. □

Note that taking $\mathrm{W}^{(\lambda,\gamma)}(u,v,w) = u^{\lambda-\frac{1}{2}}v^{\lambda-\frac{1}{2}}(1-w)^{\gamma}$ enables us to separate the integrand in the proof of Theorem 4.1. Also note that taking $2\lambda+\gamma = 1$ enables us to use Lemma 3.4 in the proof of Theorem 4.1.

In the following theorem, we show that $\mathscr{C}_{n,r}^{(\lambda,\gamma)}(u,v,w)$ is orthogonal to each polynomial of degree n. And thus the bivariate polynomials $\mathscr{C}_{n,r}^{(\lambda,\gamma)}(u,v,w)$, $r = 0, 1, \ldots, n$, and $n = 0, 1, 2, \ldots$ form an orthogonal system over the triangular domain T with respect to the weight function $\mathrm{W}^{(\lambda,\gamma)}(u,v,w)$, $\lambda > -\frac{1}{2}, \gamma > -1$.

Theorem 4.2. *For $r \neq s$, we have $\mathscr{C}_{n,r}^{(\lambda,\gamma)}(u,v,w) \perp \mathscr{C}_{n,s}^{(\lambda,\gamma)}(u,v,w)$ with respect to the weight function $\mathrm{W}^{(\lambda,\gamma)}(u,v,w) = u^{\lambda-\frac{1}{2}}v^{\lambda-\frac{1}{2}}(1-w)^{\gamma}$ such that $\lambda > -\frac{1}{2}, \gamma > -1$.*

Proof. For $r \neq s$, we have

$$I := \iint_T \mathscr{C}_{n,r}^{(\lambda,\gamma)}(u,v,w)\mathscr{C}_{n,s}^{(\lambda,\gamma)}(u,v,w)\mathrm{W}^{(\lambda,\gamma)}(u,v,w)dA$$

$$= \Delta \int_0^1 \int_0^{1-w} C_r^{(\lambda)}(\frac{u}{1-w})C_s^{(\lambda)}(\frac{u}{1-w})(1-w)^{r+s}q_{n,r}(w)q_{n,s}(w)\mathrm{W}^{(\lambda,\gamma)}(u,v,w)dudw.$$

By making the substitution $t = \frac{u}{1-w}$, we have

$$I = \Delta \int_0^1 C_r^{(\lambda)}(t)C_s^{(\lambda)}(t)t^{\lambda-\frac{1}{2}}\,(1-t)^{\lambda-\frac{1}{2}}\,dt \int_0^1 q_{n,r}(w)q_{n,s}(w)(1-w)^{2\lambda+\gamma+r+s}dw,$$

the first integral equals zero by orthogonality property of the Ultraspherical polynomials for $r \neq s$, and thus the theorem follows. □

5 Ultraspherical in Bernstein Basis

The Bernstein-Bézier form of curves and surfaces exhibits some interesting geometric properties, see [6, 9]. So, we write the orthogonal polynomials $\mathscr{C}_{n,r}^{(\lambda,\gamma)}(u,v,w)$, $r = 0,1,\ldots,n$ and $n = 0,1,2,\ldots$ in the following Bernstein-Bézier form:

$$\mathscr{C}_{n,r}^{(\lambda,\gamma)}(u,v,w) = \sum_{|\zeta|=n} a_\zeta^{n,r} b_\zeta^n(u,v,w). \tag{13}$$

We are interested in finding a closed form for the computation of the Bernstein coefficients $a_\zeta^{n,r}$. These are given explicitly in the following theorem:

Theorem 5.1. *The Bernstein coefficients $a_\zeta^{n,r}$ of equation (13) are given explicitly by*

$$a_{ijk}^{n,r} = \begin{cases} (-1)^k \frac{\binom{n+r+1}{k}\binom{n-r}{k}}{\binom{n}{k}} \mu_{i,r}^{n-k} & 0 \leq k \leq n-r \\ 0 & k > n-r \end{cases} \tag{14}$$

where $\mu_{i,r}^{n-k}$ are given in (4).

Proof. From equation (6), it is clear that $\mathscr{C}_{n,r}^{(\lambda,\gamma)}(u,v,w)$ has degree $\leq n-r$ in the variable w, and thus

$$a_{ijk}^{n,r} = 0 \quad for\ k > n-r. \tag{15}$$

For $0 \leq k \leq n-r$, the remaining coefficients are determined by equating (6) and (13) as follows

$$\sum_{i+j=n-k} a_{ijk}^{n,r} b_{ijk}^{n}(u,v,w) = (-1)^k \binom{n+r+1}{k} b_k^{n-r}(w,u+v) \sum_{i=0}^{r} c(i,\lambda) b_i^r(u,v).$$

Comparing powers of w on both sides, we have

$$\sum_{i=0}^{n-k} a_{ijk}^{n,r} \frac{n!}{i!j!k!} u^i v^j = (-1)^k \binom{n+r+1}{k}\binom{n-r}{k}(u+v)^{n-r-k} \sum_{i=0}^{r} c(i,\lambda) b_i^r(u,v).$$

The left hand side of the last equation can be written in the form

$$\sum_{i=0}^{n-k} a_{ijk}^{n,r} \frac{n!(n-k)!}{i!(n-k-i)!k!(n-k)!} u^i v^j = \sum_{i=0}^{n-k} a_{ijk}^{n,r} \binom{n}{k} b_i^{n-k}(u,v)$$

Now, we get

$$\sum_{i=0}^{n-k} a_{ijk}^{n,r} \binom{n}{k} b_i^{n-k}(u,v) = (-1)^k \binom{n+r+1}{k}\binom{n-r}{k}(u+v)^{n-r-k} \sum_{i=0}^{r} c(i,\lambda) b_i^r(u,v).$$

With some binomial simplifications, and using Lemma 3.2, we get

$$\sum_{i=0}^{n-k} a_{ijk}^{n,r} \binom{n}{k} b_i^{n-k}(u,v) = (-1)^k \binom{n+r+1}{k}\binom{n-r}{k} \sum_{i=0}^{r} \mu_{i,r}^{n-k} b_i^{n-k}(u,v), \tag{16}$$

where $\mu_{i,r}^{n-k}$ are the coefficients resulting from writing Ultraspherical polynomial of degree r in the Bernstein basis of degree $n-k$, as defined by expression (4). Thus, the required Bernstein-Bézier coefficients are given by:

$$a_{ijk}^{n,r} = \begin{cases} (-1)^k \frac{\binom{n+r+1}{k}\binom{n-r}{k}}{\binom{n}{k}} \mu_{i,r}^{n-k} & 0 \leq k \leq n-r \\ 0 & k > n-r \end{cases}.$$

□

To derive a recurrence relation for the coefficients $a_{ijk}^{n,r}$ of $\mathscr{C}_{n,r}^{(\lambda,\gamma)}(u,v,w)$, consider the generalized Bernstein polynomial of degree $n-1$,

$$\begin{aligned} b_{ijk}^{n-1}(u,v,w) &= \frac{(n-1)!}{i!j!k!} u^i v^j w^k (u+v+w) \\ &= \frac{(i+1)}{n} b_{i+1,j,k}^{n}(u,v,w) + \frac{(j+1)}{n} b_{i,j+1,k}^{n}(u,v,w) + \frac{(k+1)}{n} b_{i,j,k+1}^{n}(u,v,w) \end{aligned}$$

by construction of $\mathscr{C}_{n,r}^{(\lambda,\gamma)}(u,v,w)$, we have $\langle b_{ijk}^{n-1}(u,v,w), \mathscr{C}_{n,r}^{(\lambda,\gamma)}(u,v,w)\rangle = 0$, $i+j+k=n-1$, and thus by Lemma 2.2, we obtain

$$(i+1)a_{i+1,j,k}^{n,r} + (j+1)a_{i,j+1,k}^{n,r} + (k+1)a_{i,j,k+1}^{n,r} = 0. \tag{17}$$

But, form Theorem 5.1, we have

$$a_{i,n-i,0}^{n,r} = \mu_{i,r}^{n} \text{ for } i = 0,1,\ldots,n; \tag{18}$$

we can use (17) to generate $a_{i,j,k}^{n,r}$ recursively on k.

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