International Journal of Pure and Applied Mathematics
Volume 99 No. 3 2015, 343-354
ISSN: 1311-8080 (printed version); ISSN: 1314-3395 (on-line version)
url: http://www.ijpam.eu
doi: http://dx.doi.org/10.12732/ijpam.v99i3.9
ijpam.eu

CONSTRUCTION OF TCHEBYSHEV-II WEIGHTED ORTHOGONAL POLYNOMIALS ON TRIANGULAR

Mohammad A. AlQudah
Department of Mathematics
Northwood University
Midland, MI 48640 USA

Abstract: We construct Tchebyshev-II (second kind) weighted orthogonal polynomials $\mathscr{U}_{n,r}^{(\gamma)}(u,v,w)$, $\gamma > -1$, on the triangular domain T. We show that $\mathscr{U}_{n,r}^{(\gamma)}(u,v,w)$, $n = 0,1,2,\dots$, $r = 0,1,\dots,n$, form an orthogonal system over T with respect to the Tchebyshev-II weight function.

AMS Subject Classification: 42C05, 33C45, 33C70
Key Words: Tchebyshev, orthogonal, triangular, Bernstein, bivariate

1. Introduction

In the last couple decades, orthogonal polynomials have been studied thoroughly [8] and [12]. The Tchebyshev orthogonal polynomial of the second kind (Tchebyshev-II) is among these orthogonal polynomials. Although the main definitions and basic properties were defined many years ago, see [3] and [11], the cases of bivariate or more variables are limited.

Tchebyshev-II polynomials $U_{n,r}^{(\gamma)}(u,v,w)$ are orthogonal to each polynomial of degree $\leq n-1$, with respect to the weight function $\mathrm{W}^{(\gamma)}(u,v,w) = u^{\frac{1}{2}}v^{\frac{1}{2}}(1-w)^{\gamma}$, $\gamma > -1$ on triangular domain T as defined in [1] and [8]. However, for $r \neq s$, $U_{n,r}^{(\gamma)}(u,v,w)$ and $U_{n,s}^{(\gamma)}(u,v,w)$ are not orthogonal with respect to the weight function $\mathrm{W}^{(\gamma)}(u,v,w)$ on T.

Received: November 18, 2014

In this paper, we construct bivariate orthogonal polynomials $\mathscr{U}_{n,r}^{(\gamma)}(u,v,w)$, $r = 0,1,\ldots,n$; $n = 0,1,2,\ldots$, with respect to the Tchebyshev-II weight function $\mathrm{W}^{(\gamma)}(u,v,w) = u^{\frac{1}{2}}v^{\frac{1}{2}}(1-w)^{\gamma}, \gamma > -1$, on triangular domain T. We show that $\mathscr{U}_{n,r}^{(\gamma)}(u,v,w)$ form an orthogonal system over the triangular domain T with respect to the weight function $\mathrm{W}^{(\gamma)}(u,v,w) = u^{\frac{1}{2}}v^{\frac{1}{2}}(1-w)^{\gamma}, \gamma > -1$. Worth to mention that these Tchebyshev-II weighted orthogonal polynomials are given in the Bernstein basis form; they preserve many geometric properties of the Bernstein polynomial basis.

The construction of bivariate orthogonal polynomials on the square G is straightforward [11], where $G = \{(x,y) : -1 \leq x \leq 1, -1 \leq y \leq 1\}$. It can be done by considering the tensor product of the set of orthogonal polynomials over G.

Let $\{U_n(x)\}$ be the Tchebyshev-II polynomials over $[-1,1]$ with respect to the weight function $\mathrm{W}_1(x) = (1-x^2)^{\frac{1}{2}}$, and $\{Q_m(y)\}$ be the Tchebyshev-II polynomials over $[-1,1]$ with respect to the weight function $\mathrm{W}_2(y) = (1-y^2)^{\frac{1}{2}}$. The bivariate polynomials $\{R_{nm}(x,y)\}$ on G formed by the tensor products of the Tchebyshev-II polynomials defined as

$$R_{nm}(x,y) := U_{n-m}(x)Q_m(y), n = 0,1,2,\ldots, m = 0,1,\ldots,n.$$

The bivariate polynomials $\{R_{nm}(x,y)\}$ are orthogonal on the square G with respect to the weight function $\mathrm{W}(x,y) = \mathrm{W}_1(x)\mathrm{W}_2(y)$. However, the construction of orthogonal polynomials over a triangular domains are not straightforward like the tensor product over the square G.

2. Bernstein and Orthogonal Polynomials over Triangular Domains

Consider a base triangle in the plane with the vertices $\mathbf{p}_k = (x_k, y_k)$, $k = 1,2,3$. Then every point $\mathbf{p}$ inside the triangle T can be written using the barycentric coordinates (u,v,w) as $\mathbf{p}=u\mathbf{p}_1 + v\mathbf{p}_2 + w\mathbf{p}_3$, where $u,v,w \geq 0$, $u+v+w = 1$. The barycentric coordinates are the ratio of areas of subtriangles of the base triangle as follows:

$$u = \frac{area(\mathbf{p},\mathbf{p}_2,\mathbf{p}_3)}{area(\mathbf{p}_1,\mathbf{p}_2,\mathbf{p}_3)}, \quad v = \frac{area(\mathbf{p}_1,\mathbf{p},\mathbf{p}_3)}{area(\mathbf{p}_1,\mathbf{p}_2,\mathbf{p}_3)}, \quad w = \frac{area(\mathbf{p}_1,\mathbf{p}_2,\mathbf{p})}{area(\mathbf{p}_1,\mathbf{p}_2,\mathbf{p}_3)}. \tag{1}$$

where $\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3$ are not collinear.

Definition 2.1. The Bernstein polynomials $b_i^n(u), u \in [0,1], i = 0,1,\ldots,n$, are defined by:

$$b_i^n(u) = \begin{cases} \binom{n}{i} u^i (1-u)^{n-i} & i = 0,1,\ldots,n \\ 0 & else \end{cases} \tag{2}$$

where $\binom{n}{i}$ are the binomial coefficients.

Let $\zeta = (i,j,k)$ denote triples of nonnegative integers, where $|\zeta| = i+j+k$. The generalized Bernstein polynomials of degree n on the triangular domain

$$T = \{(u,v,w) : u,v,w \geq 0, u+v+w = 1\},$$

are defined by

$$b_\zeta^n(u,v,w) = \binom{n}{\zeta} u^i v^j w^k, \quad \text{where} \quad |\zeta| = n \quad \text{and} \quad \binom{n}{\zeta} = \frac{n!}{i!j!k!}.$$

Note that the generalized Bernstein polynomials are nonnegative over T and form a partition of unity,

$$1 = (u+v+w)^n = \sum_{\substack{0 \leq i,j,k \leq n \\ i+j+k=n}} \frac{n!}{i!j!k!} u^i v^j w^k. \tag{3}$$

These polynomials define the Bernstein basis for the space Π_n, the space of all polynomials of degree n over the triangular domain T.

A basis of linearly independent and mutually orthogonal polynomials in the barycentric coordinates (u,v,w) are constructed over T. These polynomials are represented in the following triangular table

$$\begin{array}{lllll} b_{0,0} & & & & \\ b_{1,0} & b_{1,1} & & & \\ b_{2,0} & b_{2,1} & b_{2,2} & & \\ & \vdots & & & \\ b_{n,0} & b_{n,1} & b_{n,2} & \ldots & b_{n,n}. \end{array}$$

The kth row of this table contains $k+1$ polynomials. Thus, there are $\frac{(n+1)(n+2)}{2}$ polynomials in a basis of linearly independent polynomials of total degree n. Therefore, the sum (3) involves a total of $\frac{(n+1)(n+2)}{2}$ linearly independent polynomials.

Any polynomial $p(u,v,w)$ of degree n can be written in the Bernstein form

$$p(u,v,w) = \sum_{|\zeta|=n} d_\zeta b_\zeta^n(u,v,w), \tag{4}$$

with Bézier coefficients d_ζ. We can use the degree elevation algorithm for the Bernstein representation (4) by multiplying both sides by $1 = u + v + w$ and writing

$$p(u,v,w) = \sum_{|\zeta|=n+1} d_\zeta^{(1)} b_\zeta^{n+1}(u,v,w),$$

where the the coefficients $d_\zeta^{(1)}$ are defined in [4] and [7] as

$$d_{i,j,k}^{(1)} = \frac{1}{n+1}(id_{i-1,j,k} + jd_{i,j-1,k} + kd_{i,j,k-1}), \quad i+j+k = n+1.$$

Lemma 2.1. *[5] The Bernstein polynomials $b_\zeta^n(u,v,w), |\zeta| = n$, on T satisfy*

$$\iint_T b_\zeta^n(u,v,w)dA = \frac{\Delta}{(n+1)(n+2)},$$

where Δ is double the area of T.

Definition 2.2. Let $p(u,v,w)$ and $q(u,v,w)$ be two bivariate polynomials over T, then their inner product over T defined by

$$\langle p, q\rangle = \frac{1}{\Delta}\iint_T pq dA, \quad \text{where } p \text{ and } q \text{ are orthogonal if } \langle p, q\rangle = 0.$$

For $m \geq 1$, $\mathfrak{L}_\mathfrak{m} = \{p \in \Pi_m : p \perp \Pi_{m-1}\}$ is the space of polynomials of degree m that are orthogonal to all polynomials of degree $< m$ over a triangular domain T.

Let $f(u,v,w)$ be an integrable function over T and consider the operator

$$S_n(f) = (n+1)(n+2)\sum_{|\zeta|=n} \left\langle f, b_\zeta^n\right\rangle b_\zeta^n.$$

For $n \geq m$, $\lambda_{m,n} = \frac{(n+2)!n!}{(n+m+2)!(n-m)!}$ is an eigenvalue of the operator S_n, and $\mathfrak{L}_\mathfrak{m}$ is the corresponding eigenspace [2]. The following two lemmas will be used in the proof of the main results.

Lemma 2.2. *[5] Let $p = \sum_{|\zeta|=n} c_\zeta b_\zeta^n \in \mathfrak{L}_\mathfrak{m}$ and let $q = \sum_{|\zeta|=n} d_\zeta b_\zeta^n \in \Pi_n$ with $m \leq n$. Then,*

$$\langle p, q\rangle = \frac{(n!)^2}{(n+m+2)!(n-m)!}\sum_{|\zeta|=n} c_\zeta d_\zeta.$$

Lemma 2.3. *[5] Let* $p = \sum_{|\zeta|=n} c_\zeta b_\zeta^n \in \Pi_n$. *Then,*

$$p \in \mathfrak{L}_\mathfrak{n} \iff \sum_{|\zeta|=n} c_\zeta d_\zeta = 0 \;\; \forall q = \sum_{|\zeta|=n} d_\zeta b_\zeta^n \in \Pi_{n-1}. \tag{5}$$

For the main results simplifications, we define the double factorial of an integer n as

$$\begin{aligned} (2n-1)!! &= (2n-1)(2n-3)(2n-5)\dots(3)(1) \quad &\text{if } n \text{ is odd} \\ n!! &= (n)(n-2)(n-4)\dots(4)(2) \quad &\text{if } n \text{ is even} \end{aligned} \tag{6}$$

where $0!! = (-1)!! = 1$.

3. Tchebyshev-II Weighted Orthogonal Polynomials

Tchebyshev-II polynomials $U_n(x)$ of degree n are the orthogonal polynomials except for a constant factor on $[-1,1]$ with respect to the weight function $\mathrm{W}(x) = (1-x^2)^{\frac{1}{2}}$. For simplicity, without loss of generality, we take $x \in [0,1]$ for both Bernstein and Tchebyshev-II polynomials.

The following lemmas will be needed in the construction of the orthogonal bivariate polynomials and the proof of the main results.

Lemma 3.1. *[10] The Tchebyshev-II polynomials* $U_r(x)$ *have the Bernstein representation:*

$$U_r(x) = \frac{(r+1)(2r)!!}{(2r+1)!!} \sum_{i=0}^{r} (-1)^{r-i} \frac{\binom{r+\frac{1}{2}}{i}\binom{r+\frac{1}{2}}{r-i}}{\binom{r}{i}} b_i^r(x), \;\; r = 0,1,\dots. \tag{7}$$

Lemma 3.2. *[10] The Tchebyshev-II polynomials* $U_0(x),\dots,U_n(x)$ *of degree* $\le n$ *can be expressed in the Bernstein basis of fixed degree* n *by the following formula*

$$U_r(x) = \sum_{i=0}^{n} \mu_{i,r}^n b_i^n(x), \quad r = 0,1,\dots,n$$

where

$$\mu_{i,r}^n = \frac{(r+1)(2r)!!}{(2r+1)!!}\binom{n}{i}^{-1} \sum_{k=\max(0,i+r-n)}^{\min(i,r)} (-1)^{r-k}\binom{n-r}{i-k}\binom{r+\frac{1}{2}}{k}\binom{r+\frac{1}{2}}{r-k} \tag{8}$$

Using Pochhammer symbol is more appropriate in (3.1) and (8), but the combinatorial notation gives more compact and readable formulas, these have also been used by Szegö [12].

In the following lemma, let

$$q_{n,r}(w) = \sum_{j=0}^{n-r} (-1)^j \binom{n+r+1}{j} b_j^{n-r}(w). \tag{9}$$

The polynomial $q_{n,r}(w)$ is a scalar multiple of $U_{n-r}(1-2w)$.

Lemma 3.3. *[5] For $r = 0, \ldots, n$ and $i = 0, \ldots, n-r-1$, $q_{n,r}(w)$ is orthogonal to $(1-w)^{2r+i+1}$ on $[0,1]$. Hence for every polynomial $p(w)$ of degree $\leq n-r-1$, we have*

$$\int_0^1 q_{n,r}(w)p(w)(1-w)^{2r+1}dw = 0.$$

Analogous to [5], a simple closed-form representation of degree-ordered system of orthogonal polynomials is constructed on a triangular domain T using Bernstein polynomials, since Bernstein polynomials are stable [6].

For $r = 0, 1, \ldots, n$ and $n = 0, 1, 2, \ldots$, we define the bivariate polynomials

$$\mathscr{U}_{n,r}^{(\gamma)}(u,v,w) = \sum_{i=0}^{r} c(i) b_i^r(u,v) \sum_{j=0}^{n-r} (-1)^j \binom{n+r+1}{j} b_j^{n-r}(w, u+v), \tag{10}$$

where $\gamma > -1$, $b_i^r(u,v)$ defined in (2) and

$$c(i) = (-1)^{r-i} \frac{\binom{r+\frac{1}{2}}{i}\binom{r+\frac{1}{2}}{r-i}}{\binom{r}{i}}, \quad i = 0, 1, \ldots, r. \tag{11}$$

By choosing $\mathscr{U}_{0,0}^{(\gamma)} = 1$, the polynomials $\mathscr{U}_{n,r}^{(\gamma)}(u,v,w)$ for $0 \leq r \leq n$ and $n = 0, 1, 2, \ldots$ form a degree-ordered orthogonal sequence over T.

Rewriting (10) using Tchebyshev-II polynomials form, we obtain

$$\mathscr{U}_{n,r}^{(\gamma)}(u,v,w) = \sum_{i=0}^{r} c(i) \frac{b_i^r(u,v)}{(u+v)^r} (1-w)^r \sum_{j=0}^{n-r} (-1)^j \binom{n+r+1}{j} b_j^{n-r}(w, 1-w).$$

Using Lemma 3.1 and $\frac{b_i^r(u,v)}{(u+v)^r} = b_i^r(\frac{u}{1-w})$, and we get

$$\mathscr{U}_{n,r}^{(\gamma)}(u,v,w) = \frac{\binom{r+\frac{1}{2}}{r}}{(r+1)} U_r(\frac{u}{1-w})(1-w)^r q_{n,r}(w), \quad r = 0, \ldots, n, \tag{12}$$

where $U_r(t)$ is the univariate Tchebyshev-II polynomial of degree r and $q_{n,r}(w)$ is defined in equation (9).

For simplicity, we rewrite (12) as

$$\mathscr{U}_{n,r}^{(\gamma)}(u,v,w) = U_r(\frac{u}{1-w})(1-w)^r q_{n,r}(w), \quad r = 0,\dots,n, \tag{13}$$

since we are dealing with orthogonality, and the Tchebyshev-II polynomials $U_n(x)$ of degree n are the orthogonal except for a constant factor.

The polynomials $\mathscr{U}_{n,r}^{(\gamma)}(u,v,w)$ form an orthogonal system if $\mathscr{U}_{n,r}^{(\gamma)}(u,v,w) \in \mathfrak{L}_{\mathfrak{n}}$, $n \geq 1$, $r = 0,1,\dots,n$, and for $r \neq s$ $\mathscr{U}_{n,r}^{(\gamma)}(u,v,w) \perp \mathscr{U}_{n,s}^{(\gamma)}(u,v,w)$. In the following theorem, we show that the polynomials $\mathscr{U}_{n,r}^{(\gamma)}(u,v,w)$, $r = 0,\dots,n$, are orthogonal to all polynomials of degree less than n over the triangular domain T.

Theorem 3.1. *For each $r = 0,1,\dots,n$ and $n = 1,2,\dots$, $\mathscr{U}_{n,r}^{(\gamma)}(u,v,w) \in \mathfrak{L}_n$ with respect to the weight function* $\mathrm{W}^{(\gamma)}(u,v,w) = u^{\frac{1}{2}}v^{\frac{1}{2}}(1-w)^{\gamma}$, *where* $\gamma > -1$.

Proof. Let

$$Q_{s,m}(u,v,w) = U_s(\frac{u}{1-w})(1-w)^m w^{n-m-1},\ m = 0,\dots,n-1, s = 0,\dots,m, \tag{14}$$

be the set of bivariate polynomials. The span of (14) includes the set of Bernstein polynomials

$$\begin{aligned} b_j^m(\frac{u}{1-w})(1-w)^m w^{n-m-1} &= b_j^m(u,v)(1-w)^m w^{n-m-1}\frac{1}{(1-w)^m} \\ &= b_j^m(u,v)w^{n-m-1}, \quad j = 0,\dots,m; m = 0,\dots,n-1, \end{aligned}$$

which span Π_{n-1}.

It is sufficient to show that for each $s = 0,\dots,m$; $m = 0,\dots,n-1$,

$$I := \iint_T \mathscr{U}_{n,r}^{(\gamma)}(u,v,w)Q_{s,m}(u,v,w)\mathrm{W}^{(\gamma)}(u,v,w)dA = 0. \tag{15}$$

The integral (15) can be simplified to

$$I = \Delta \int_0^1 \int_0^{1-w} U_r(\frac{u}{1-w})q_{n,r}(w)U_s(\frac{u}{1-w})w^{n-m-1}u^{\frac{1}{2}}v^{\frac{1}{2}}(1-w)^{\gamma+r+m}dudw. \tag{16}$$

Using the substitution $t = \frac{u}{1-w}$ in (16) we have

$$I = \Delta \int_0^1 \int_0^1 U_r(t) q_{n,r}(w) U_s(t)(1-w)^{\gamma+r+m+2} w^{n-m-1} t^{\frac{1}{2}} (1-t)^{\frac{1}{2}} \, dt dw$$
$$= \Delta \int_0^1 U_r(t) U_s(t) t^{\frac{1}{2}} (1-t)^{\frac{1}{2}} \, dt \int_0^1 q_{n,r}(w)(1-w)^{\gamma+r+m+2} w^{n-m-1} dw.$$

If $m < r$, then $s < r$, the first integral is zero by the orthogonality property of the Tchebyshev-II polynomials. If $r \le m \le n-1$, then by Lemma 3.3 the second integral equals zero. Thus the theorem follows. □

Note that taking $\mathrm{W}^{(\gamma)}(u,v,w) = u^{\frac{1}{2}} v^{\frac{1}{2}} (1-w)^{\gamma}$ enables us to separate the integrand in the proof of Theorem 3.1. Also taking $\gamma > -1$ enables us to use Lemma 3.3 in the proof of Theorem 3.1.

In the following theorem, we show that $\mathscr{U}_{n,r}^{(\gamma)}(u,v,w)$ is orthogonal to each polynomial of degree n.

Theorem 3.2. *For $r \neq s$, $\mathscr{U}_{n,r}^{(\gamma)}(u,v,w) \perp \mathscr{U}_{n,s}^{(\gamma)}(u,v,w)$ with respect to the weight function $\mathrm{W}^{(\gamma)}(u,v,w) = u^{\frac{1}{2}} v^{\frac{1}{2}} (1-w)^{\gamma}$ where $\gamma > -1$.*

Proof. For $r \neq s$, we have

$$I := \iint_T \mathscr{U}_{n,r}^{(\gamma)}(u,v,w) \mathscr{U}_{n,s}^{(\gamma)}(u,v,w) \mathrm{W}^{(\gamma)}(u,v,w) dA$$
$$= \Delta \int_0^1 \int_0^{1-w} U_r\left(\frac{u}{1-w}\right) U_s\left(\frac{u}{1-w}\right) (1-w)^{r+s} q_{n,r}(w) q_{n,s}(w) \, \mathrm{W}^{(\gamma)}(u,v,w) du dw.$$

Using the substitution $t = \frac{u}{1-w}$, we get

$$I = \Delta \int_0^1 U_r(t) U_s(t) t^{\frac{1}{2}} (1-t)^{\frac{1}{2}} \, dt \int_0^1 q_{n,r}(w) q_{n,s}(w) (1-w)^{\gamma+r+s+2} dw.$$

the first integral equals zero by orthogonality property of the Tchebyshev-II polynomials for $r \neq s$, and thus the theorem follows. □

4. Orthogonal Polynomials in Bernstein Basis

The Bernstein-Bézier form of curves and surfaces exhibits some interesting geometric properties, see [4] and [7]. Writing the orthogonal polynomials $\mathscr{U}_{n,r}^{(\gamma)}(u, v, w)$, $r = 0, 1, \dots, n$ and $n = 0, 1, 2, \dots$ in the following Bernstein-Bézier form:

$$\mathscr{U}_{n,r}^{(\gamma)}(u, v, w) = \sum_{|\zeta|=n} a_{\zeta}^{n,r} b_{\zeta}^{n}(u, v, w). \tag{17}$$

The following theorem provides a closed form of the Bernstein coefficients $a_{\zeta}^{n,r}$.

Theorem 4.1. *The Bernstein coefficients $a_{\zeta}^{n,r}$ are given by*

$$a_{ijk}^{n,r} = \begin{cases} (-1)^k \dfrac{\binom{n+r+1}{k}\binom{n-r}{k}}{\binom{n}{k}} \mu_{i,r}^{n-k} & 0 \le k \le n-r \\ 0 & k > n-r \end{cases}, \tag{18}$$

where $\mu_{i,r}^{n-k}$ are given in (8).

Proof. From equation (10), it is clear that $\mathscr{U}_{n,r}^{(\gamma)}(u, v, w)$ has degree $\le n-r$ in the variable w, thus

$$a_{ijk}^{n,r} = 0 \quad for\ k > n-r. \tag{19}$$

For $0 \le k \le n-r$, the remaining coefficients are determined by equating (10) and (17) as follows

$$\sum_{i+j=n-k} a_{ijk}^{n,r} b_{ijk}^{n}(u, v, w) = (-1)^k \binom{n+r+1}{k} b_k^{n-r}(w, u+v) \sum_{i=0}^{r} c(i) b_i^r(u, v).$$

Comparing powers of w on both sides, we have

$$\sum_{i=0}^{n-k} a_{ijk}^{n,r} \frac{n!}{i!j!k!} u^i v^j = (-1)^k \binom{n+r+1}{k} \binom{n-r}{k} (u+v)^{n-r-k} \sum_{i=0}^{r} c(i) b_i^r(u, v).$$

The left hand side of the last equation can be written in the form

$$\sum_{i=0}^{n-k} a_{ijk}^{n,r} \frac{n!}{i!j!k!} u^i v^j = \sum_{i=0}^{n-k} a_{ijk}^{n,r} \frac{n!(n-k)!}{i!(n-k-i)!k!(n-k)!} u^i v^j$$
$$= \sum_{i=0}^{n-k} a_{ijk}^{n,r} \binom{n}{k} b_i^{n-k}(u, v).$$

Therefore,

$$\sum_{i=0}^{n-k} a_{ijk}^{n,r} \binom{n}{k} b_i^{n-k}(u,v)$$
$$= (-1)^k \binom{n+r+1}{k} \binom{n-r}{k} (u+v)^{n-r-k} \sum_{i=0}^{r} c(i) b_i^r(u,v).$$

Using Lemma 3.2 with some binomial simplifications, we get

$$\sum_{i=0}^{n-k} a_{ijk}^{n,r} \binom{n}{k} b_i^{n-k}(u,v) = (-1)^k \binom{n+r+1}{k} \binom{n-r}{k} \sum_{i=0}^{r} \mu_{i,r}^{n-k} b_i^{n-k}(u,v), \quad (20)$$

where $\mu_{i,r}^{n-k}$ are the coefficients resulting from writing Tchebyshev-II polynomial of degree r in the Bernstein basis of degree $n-k$, as defined by expression (8). The result in (18) follows. □

To derive a recurrence relation for the coefficients $a_{ijk}^{n,r}$ of $\mathscr{U}_{n,r}^{(\gamma)}(u,v,w)$, consider the generalized Bernstein polynomial of degree $n-1$,

$$\begin{aligned} b_{ijk}^{n-1}(u,v,w) &= \frac{(n-1)!}{i!j!k!} u^i v^j w^k (u+v+w) \\ &= \frac{(i+1)}{n} b_{i+1,j,k}^n(u,v,w) + \frac{(j+1)}{n} b_{i,j+1,k}^n(u,v,w) \\ &\quad + \frac{(k+1)}{n} b_{i,j,k+1}^n(u,v,w). \end{aligned}$$

By the construction of $\mathscr{U}_{n,r}^{(\gamma)}(u,v,w)$, we have

$$\langle b_{ijk}^{n-1}(u,v,w), \mathscr{U}_{n,r}^{(\gamma)}(u,v,w) \rangle = 0, \quad i+j+k = n-1.$$

Thus by Lemma 2.3

$$(i+1) a_{i+1,j,k}^{n,r} + (j+1) a_{i,j+1,k}^{n,r} + (k+1) a_{i,j,k+1}^{n,r} = 0. \quad (21)$$

From Theorem 4.1, $a_{i,n-i,0}^{n,r} = \mu_{i,r}^n, i = 0, 1, \ldots, n$. Therefore, we can use (21) to generate $a_{i,j,k}^{n,r}$ recursively on k.

Acknowledgments

The author thanks the referee for the comments and the suggestions.

References

[1] M. Abramowitz, I.A. Stegun (Eds.), *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables*, 9th printing. New York: Dover (1972).

[2] M.-M. Derriennic, On multivariate approximation by Bernstein-type polynomials, *J. Approx. Theory*, **45**, no. 2 (1985), 155-166.

[3] C.F. Dunkl, Y. Xu, *Orthogonal Polynomials of Several Variables, Encyclopedia of Mathematics and its Applications*, vol. 81, Cambridge University Press, Cambridge (2001).

[4] G. Farin, *Curves and Surface for Computer Aided Geometric Design*, 3rd ed., Computer Science and Scientific Computing, Academic Press, Massachusetts (1993).

[5] R.T. Farouki, T. N.T. Goodman, T. Sauer, Construction of orthogonal bases for polynomials in Bernstein form on triangular and simplex domains, *Comput. Aided Geom. Design*, **20**, no. 4 (2003), 209-230.

[6] R.T. Farouki, V.T. Rajan, On the numerical condition of polynomials in Bernstein form, *Comput. Aided Geom. Design*, **4**, no. 3 (1987), 191-216.

[7] J. Hoschek, D. Lasser, *Fundamentals of Computer Aided Geometric Design*, A K Peters, Massachusetts (1993).

[8] F.W.J. Olver, D.W. Lozier, R. F. Boisvert, and C. W. Clark, editors. *NIST handbook of mathematical functions*, Cambridge University Press, Cambridge (2010).

[9] A. Rababah, Distance for degree raising and reduction of triangular Bézier surfaces, *J. Comput. Appl. Math.*, **158**, no. 2 (2003), 233-241.

[10] A. Rababah, Transformation of ChebyshevBernstein polynomial basis, *Comput. Methods Appl. Math.*, **3** (4) (2003), pp. 608-622.

[11] P.K. Suetin, *Orthogonal Polynomials in Two Variables*, Nauka, Moscow (1999).

[12] G. Szegö, *Orthogonal Polynomials*, 4th ed., American Mathematical Society, Rhode Island (1975).